IIR FILTER

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IIR FILTER

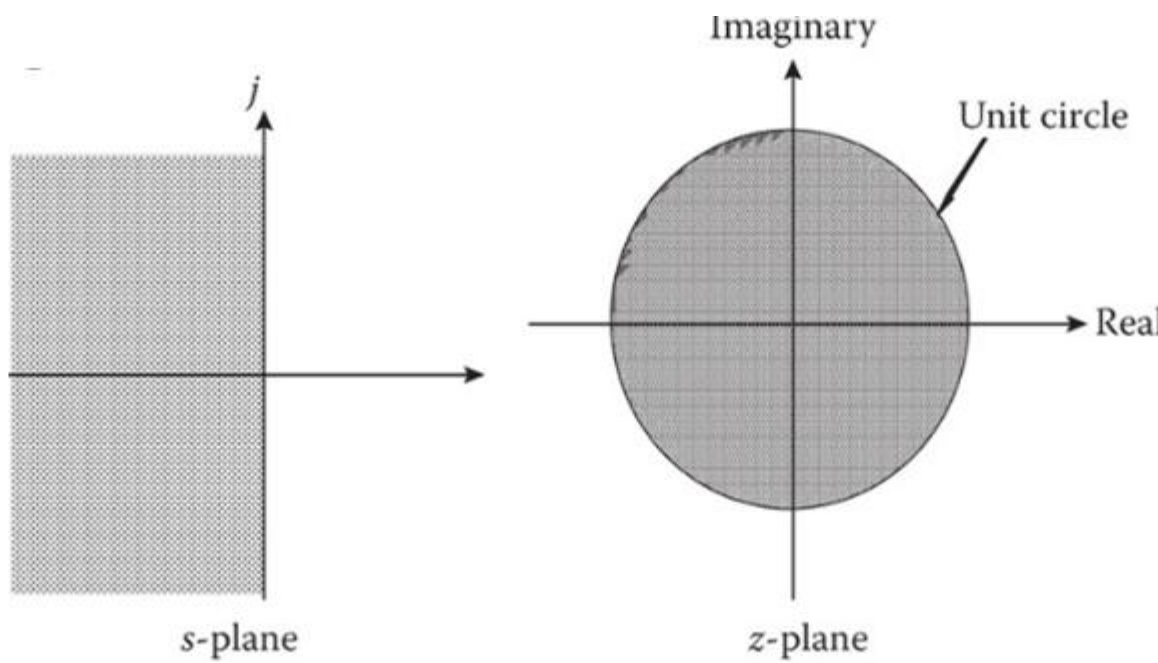
- IIR filters are of recursive type whereby the present output sample depends on the present input, past input samples and output samples.

Difference Between Analog filter and Digital filter

Sl. No.	Analog Filter	Digital Filter
1	Analog filter processes analog input and generates analog outputs.	A digital filter processes and generates digital data.
2	Analog filters are constructed from active or passive electronic components.	It consists of elements like adder, multiplier and delay unit.
3	It is described by differential equation.	It is described by difference equation.
4	The frequency response of an analog filter can be modified by changing the components.	The frequency response can be changed by changing the filter coefficients.

Procedures for digitizing the transfer function of an analog filter.

1. Approximation of derivative
2. Impulse invariant method
3. Bilinear transformation



Use the backward difference for the derivative to convert the analog LPF with system function H(s) = 1/(S+2)

Sol:

$$H(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

$$H(z) = \frac{1}{s+2}\Big|_{s=\frac{1-z^{-1}}{T}}$$

$$= \frac{1}{\frac{1-z^{-1}}{T}+2} = \frac{T}{1-z^{-1}+2T}.$$

Assume $T = 1s$

$$H(z) = \frac{1}{1-z^{-1}+2}$$

$$H(z) = \frac{1}{3-z^{-1}}$$

Use the backward difference for the derivative to convert the analog filter with system function H(s) = $1/(s^2+16)$

$$H(s) = \frac{1}{s^2+16}$$

Sol:

$$H(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{s^2+16}\Big|_{s=\frac{1-z^{-1}}{T}}$$

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{T}\right)^2+16} = \frac{T^2}{1+z^{-2}-2z^{-1}+16T^2}$$

Assume $T = 1s$

$$H(z) = \frac{1}{1+z^{-2}-2z^{-1}+16}$$

$$H(z) = \frac{1}{z^{2}-2z^{-1}+17}$$

Properties of impulse invariant method

$$\frac{1}{s-a} \rightarrow \frac{1}{1-e^{aT}z^{-1}}$$

$$\frac{1}{s+a} \rightarrow \frac{1}{1-e^{-aT}z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

For the analog transfer function $H(s) = \frac{1}{(s+1)(s+2)}$ **Determine H(z) using impulse invariant method.**

Sol:

$$H(s) = \frac{1}{(s+1)(s+2)}$$

By Partial Fraction Expansion

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

$s = -2$; $B = -1$

$s = -1$; $A = 1$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$H(z) = \frac{1}{1 - e^{-T} z^{-1}} - \frac{1}{1 - e^{-2T} z^{-1}}$$

Apply $T = 1\,sec$

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-2}z^{-1}}$$

$$= \frac{1-0.135z^{-1}-1+0.3678z^{-1}}{(1-0.3678z^{-1})(1-0.135z^{-1})}$$

$$= \frac{0.2326\,z^{-1}}{1-0.5032\,z^{-1}+0.0498\,z^{-2}}$$

$$= \frac{0.2326z^{-1}}{z^{-2}(z^2-0.5032z+0.0498)}$$

$$H(z) = \frac{0.2326\,z}{z^2-0.5032\,z+0.0498}$$

Convert the analog filter into a digital filter where transfer function is $H(s) = \frac{s+0.2}{(s+0.2)^2+9}$. Use impulse invariant method. Assume T=1sec.

Sol: Given

$$H(s) = \frac{s+0.2}{(s+0.2)^2+9} = \frac{s+0.2}{(s+0.2)^2+3^2}$$

This system function is in the form $H(s) = \frac{s+a}{(s+a)^2+b^2}$

$a = 0.2$ and $b = 3$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)\,z^{-1}}{1-2e^{-aT}(\cos bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

$$H(z) = \frac{1-e^{-0.2T}(\cos 3T)\,z^{-1}}{1-2e^{-0.2T}(\cos 3T)\,z^{-1}+e^{-0.4T}z^{-2}}$$

Sub $T = 1\,sec$

$$= \frac{1 - e^{-0.2}(\cos 3)\, z^{-1}}{1 - 2e^{-0.2}(\cos 3) z^{-1} + e^{-0.4} z^{-2}}$$

$$= \frac{1 - 0.8187(-0.99) z^{-1}}{1 - 2(0.8187)(-0.99) z^{-1} + 0.6703\, z^{-2}}$$

$$= \frac{1 + 0.8105\, z^{-1}}{1 + 0.6210 z^{-1} + 0.6703\, z^{-2}}$$

Bilinear Transformation

- The bilinear transformation is a mapping that transforms the left half of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.
- The mapping from the s-plane to z-plane in bilinear transformation is

$$s = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

- All points in the LHP of s-plane are mapped inside the unit circle in the z-plane and all points in the RHP of s-plane are mapped inside the unit circle in the z-plane

- For smaller values of ω there exist linear relationship between ω and Ω.
  that is $\omega = \Omega T$
- But for larger values of ω the relationship is non-linear. This effect is known as warping effect. This effect compresses the magnitude and phase response at high frequencies.
- The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Convert the analog filter into a digital IIR filter use bilinear transformation with system function is $\boldsymbol{H(s)=\frac{2}{(s+1)(s+2)}}$.

Assume T=0.1sec.

$$\boldsymbol{Given\ H(s)=\frac{2}{(s+1)(s+2)}}$$

$$Apply\ s=\frac{2}{T}\frac{(z-1)}{(z+1)}for\ the\ above\ eqn$$

$$H(z)=\frac{2}{(\frac{2(z-1)}{T(z+1)}+1)(\frac{2(z-1)}{T(z+1)}+2)}$$

$\boldsymbol{Given\ T=0.1sec\ we\ get}$

$$H(z)=\frac{2}{(\frac{2(z-1)}{0.1(z+1)}+1)(\frac{2(z-1)}{0.1(z+1)}+2)}$$

$$H(z) = \frac{2}{(20\frac{(z-1)}{(z+1)} + 1)(20\frac{(z-1)}{(z+1)} + 2)}$$

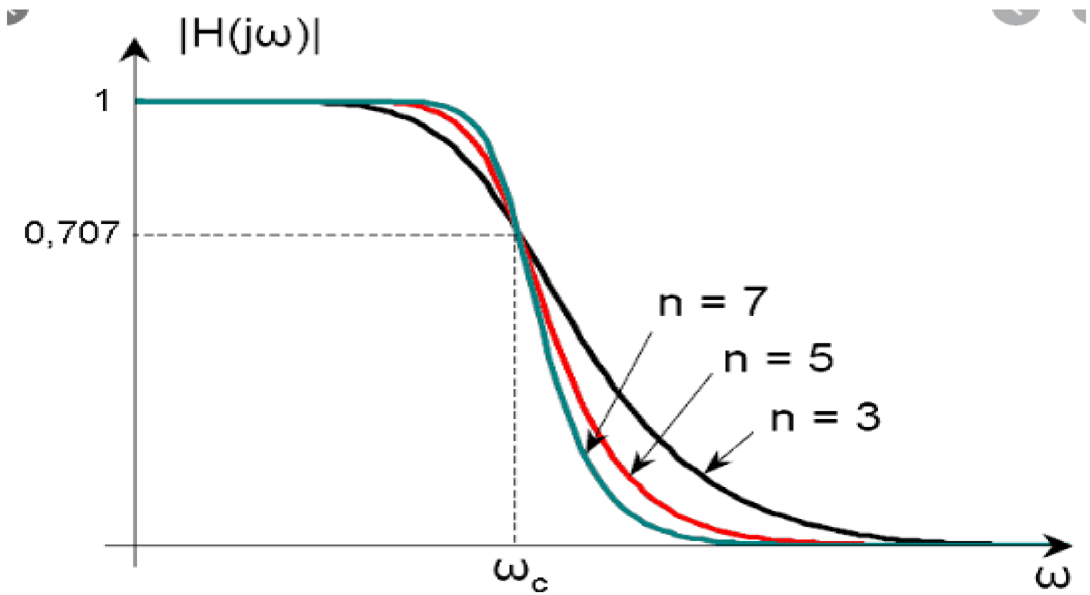
$$= \frac{2(z+1)^2}{(20z - 20 + z + 1)(20z - 20 + 3z + 3)}$$

$$= \frac{2z^2 + 4z + 2}{(21z - 19)(23z - 17)}$$

$$H(z) \quad = \frac{2z^2 + 4z + 2}{483z^3 - 794z + 323}$$

BUTTERWORTH FILTER

- The magnitude response of the Butterworth filter decreases monotonically as the frequency Ω increases from 0 to ∞.
- The magnitude response of the Butterworth filter closely approximates the ideal response as the order N increases.
- The poles of the Butterworth filter lie on circle.



Design a Butterworth filter for the following specifications

$$0.8 \le |H(e^{j\Omega})| \le 1 \quad , \quad 0 \le \omega \le 0.2\pi$$

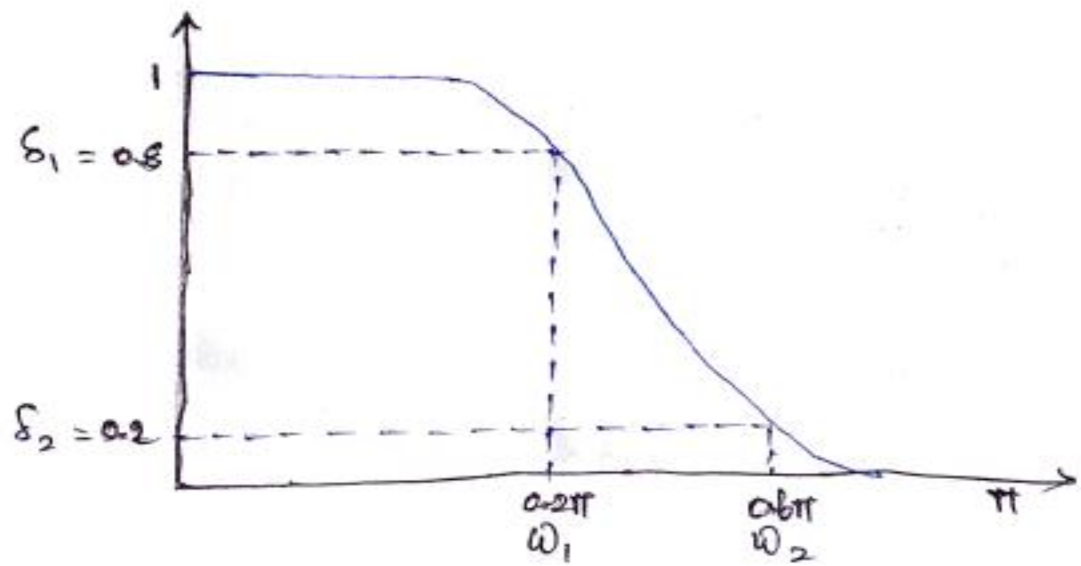
$$|H(e^{j\Omega})| \le 0.2 \quad , \quad 0.6\pi \le \omega \le \pi$$

with $T = 1\,sec$ Apply Bilinear Transformation.

Sol:

$\delta_1 = 0.8$ $\qquad \omega_1 = 0.2\pi$

$\delta_2 = 0.2$ $\qquad \omega_2 = 0.6\pi$



Step 1 : Analog filter's edge frequencies

For Bilinear Transformation

$$\Omega_1 = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.6498$$

$$\Omega_2 = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = 2\tan\left(\frac{0.6\pi}{2}\right) = 2.7527$$

Step 2 : Order of the filter

$$N \geq \frac{1}{2}\,\frac{\log\left[\left(\frac{1}{\delta_2^2}-1\right)/\left(\frac{1}{\delta_1^2}-1\right)\right]}{\log\left(\Omega_2/\Omega_1\right)}$$

$$\geq \frac{1}{2}\,\frac{\log\left[\left(\frac{1}{0.2^2}-1\right)/\left(\frac{1}{0.8^2}-1\right)\right]}{\log\left(\frac{2.7527}{0.6498}\right)}$$

$$\geq \frac{1}{2}\,\frac{\log\left(24/0.5625\right)}{0.6269}$$

$$N \geq 1.3$$

$$\boxed{N = 2}$$

Step 3 : 3dB cutoff frequency

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{\delta_1^2} - 1\right]^{1/2N}}$$

$$= \frac{0.6498}{\left(\frac{1}{0.8^2} - 1\right)^{1/4}} = \frac{0.6498}{0.5625^{0.25}}$$

$$\boxed{\Omega_c = 0.75}$$

Step 4 : Transfer function

N is even $\quad H(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$

$$H(s) = \frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}$$

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

$$b_1 = 2 \sin \frac{\pi}{4} = 1.414$$

$$B_1 = c_1 = 1$$

$$H(S) = \frac{0.75^2}{s^2 + (1.4142)(0.75)s + 0.75^2}$$

$$= \frac{0.5625}{s^2 + 1.06s + 0.5625}$$

Step 5 Digital Transfer Function

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$H(z) = \frac{0.5625}{\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)\right]^2 + 1.06\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)\right] + 0.5625}$$

$$= \frac{0.5625}{4\frac{(z-1)^2}{(z+1)^2} + 2.12\left(\frac{z-1}{z+1}\right) + 0.5625}$$

$$= \frac{0.5625(z+1)^2}{4(z-1)^2 + 2.12(z-1)(z+1) + 0.5625(z+1)^2}$$

$$= \frac{0.5625(z^2+2z+1)}{4(z^2-2z+1) + 2.12(z^2-1) + 0.5625(z^2+2z+1)}$$

$$= \frac{0.5625z^2 + 1.125z + 0.5625}{4z^2 - 8z + 4 + 2.12z^2 - 2.12 + 0.5625z^2 + 1.125z + 0.5625}$$

Design a Butterworth filter using impulse invariant method for the following specifications. Assume T= 1sec.

$$0.8 \leq |H(e^{j\Omega})| \leq 1 \quad , \quad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\Omega})| \leq 0.2 \quad , \quad 0.6\pi \leq \omega \leq \pi$$

Sol:

$\delta_1 = 0.8$ $\qquad \omega_1 = 0.2\pi$

$\delta_2 = 0.2$ $\qquad \omega_2 = 0.6\pi$

Step: 1 Analog Frequency

$$\Omega_1 = \frac{\omega_1}{T} = \frac{0.2\pi}{1} = 0.2\pi$$

$$\Omega_2 = \frac{\omega_2}{T} = \frac{0.6\pi}{1} = 0.6\pi$$

Step: 2 Order of the filter

$$N \geq \frac{1}{2} \frac{\log\left[\left(\frac{1}{\delta_2^2}-1\right)/\left(\frac{1}{\delta_1^2}-1\right)\right]}{\log(\Omega_2/\Omega_1)}$$

$$\geq \frac{1}{2} \frac{\log\left[\left(\frac{1}{0.2^2}-1\right)/\left(\frac{1}{0.8^2}-1\right)\right]}{\log(0.6\pi/0.2\pi)}$$

$$\geq \frac{1}{2} \frac{\log[24/0.5625]}{\log 3}$$

$$\geq 1.7$$

$$\boxed{N = 2}$$

Step: 3 3dB cutoff frequency

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{\delta_1^2}-1\right)^{1/2N}}$$

$$= \frac{0.2\pi}{\left(\frac{1}{0.8^2}-1\right)^{1/4}}$$

$$\boxed{\Omega_c = 0.7251}$$

Step : 4 Analog Transfer function

N is even
$$H(S) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{S^2 + b_k \Omega_c S + c_k \Omega_c^2}$$

$$= \frac{B_1 \Omega_c^2}{S^2 + b_1 \Omega_c S + c_1 \Omega_c^2}$$

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

$$b_1 = 2 \sin\left[\frac{\pi}{4}\right] = 1.414$$

$$B_1 = C_1 = 1$$

$$H(S) = \frac{0.7251^2}{S^2 + (1.414)(0.7251) S + 0.7251^2}$$

$$= \frac{0.526}{S^2 + 1.025 S + 0.526}$$

Step 5 : Digital Transfer Function

$$H(s) = \frac{0.526}{s^2+1.025s+0.526}$$

$2a = 1.025 \quad \Rightarrow \quad a = 0.5125$

$$H(s) = \frac{0.526}{(s+0.5125)^2+0.2633}$$

$$= \frac{0.526}{(s+0.5125)^2+0.5131^2}$$

$$= \frac{0.526\times 0.5131/0.5131}{(s+0.5125)^2+0.5131^2}$$

$$= 1.025 \frac{0.5131}{(s+0.5125)^2+0.5131^2}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$H(z) = 1.025 \frac{e^{-0.5125T}(\sin 0.5131T)z^{-1}}{1-2e^{-0.5125T}(\cos 0.5131T)z^{-1}+e^{-2\times 0.5125T}z^{-2}}$$

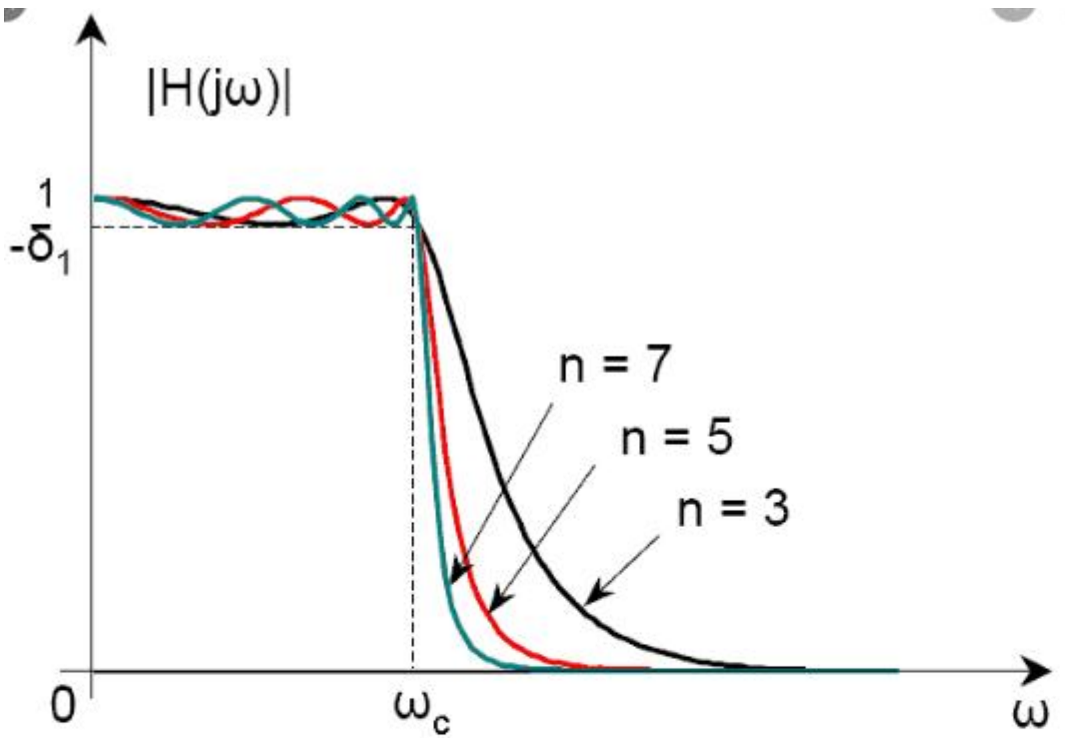
T = 1 sec

$$H(z) = \frac{1.025 \times 0.5989 \times 0.49 z^{-1}}{1 - 1.19 \times 0.871 z^{-1} + 0.358 z^{-2}}$$

$$H(z) = \frac{0.3 z^{-1}}{1 - 1.036 z^{-1} + 0.358 z^{-2}}$$

CHEBYSHEV FILTER

- The magnitude response of the Chebyshev filter exhibits ripple either in passband or in stopband according to type.
- The poles of the Chebyshev filter lie on an ellipse



Design a digital Chebyshev LPF to satisfy the constraints

$$0.707 \leq |H(e^{j\omega})| \leq 1 \quad , \quad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.1 \quad 0.5\pi \leq \omega \leq \pi$$

using Bilinear transformation and assuming T=1sec.

Sol:

Given $\delta_1 = 0.707 \qquad \omega_1 = 0.2\pi$

$\delta_2 = 0.1 \qquad \omega_2 = 0.5\pi$

Step 1. Analog Frequency

$$\Omega_1 = \frac{2}{T} \tan \frac{\omega_1}{2} = \frac{2}{1} \tan \frac{0.2\pi}{2} = 0.6498$$

$$\Omega_2 = \frac{2}{T} \tan \frac{\omega_2}{2} = \frac{2}{1} \tan \frac{0.5\pi}{2} = 2$$

$$\Omega_c = \Omega_1 = 0.6498$$

Step 2 Order of the filter

$$N \geq \frac{\cosh^{-1}\left\{\left[\left(\frac{1}{\delta_2^2}-1\right)/\left(\frac{1}{\delta_1^2}-1\right)\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

$$\geq \frac{\cosh^{-1}\left\{\left[\left(\frac{1}{0.1^2}-1\right)/\left(\frac{1}{0.707^2}-1\right)\right]^{0.5}\right\}}{\cosh^{-1}(2/0.6498)}$$

$$\geq \frac{\cosh^{-1}\left[(99/1)^{0.5}\right]}{\cosh^{-1} 3.0779}$$

$$\geq 1.669$$

$$\boxed{N = 2}$$

Step 3. Analog Transfer Function H(s)

N is even

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} = \frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{1/N} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{-1/N}\right\}$$

$$\varepsilon = \left[\frac{1}{\delta_1^2} - 1\right]^{0.5} = \left[\frac{1}{0.707^2} - 1\right]^{0.5} = 1$$

$$y_N = \frac{1}{2}\left\{2.414^{1/2} - 2.414^{-1/2}\right\} = 0.455$$

$$b_k = 2\, y_N \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

$$b_1 = 2 \times 0.455 \sin\left[\frac{\pi}{4}\right] = 0.6435$$

$$c_k = y_N^2 + \cos^2\left[\frac{(2k-1)\pi}{2N}\right]$$

$$c_1 = 0.455^2 + \cos^2\left[\frac{\pi}{4}\right] = 0.707$$

To find B_k

$$H(s)\big|_{s=0} = 1$$

$$\frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}\Bigg|_{s=0} = 1$$

$$\frac{B_1}{0.707} = 1$$

$$B_1 = 0.707$$

$$H(s) = \frac{0.6498^2 \times 0.707}{s^2 + (0.6435)(0.6498)s + 0.707 \times 0.6498^2}$$

$$= \frac{0.2985}{s^2 + 0.418s + 0.2985}$$

Step 4 Digital Transfer Function

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

T=1 sec

$$H(z) = \frac{0.2985}{2^2\left(\frac{z-1}{z+1}\right)^2 + 0.418\times 2\left(\frac{z-1}{z+1}\right) + 0.2985}$$

$$= \frac{0.2985(z+1)^2}{4(z-1)^2 + 0.836(z-1)(z+1) + 0.2985(z+1)^2}$$

$$= \frac{0.2985(z+1)^2}{4(z^2+1-2z) + 0.836(z^2-1) + 0.2985(z^2+2z+1)}$$

$$= \frac{0.2985(z^2+2z+1)}{4z^2+4-8z+0.836z^2-0.836+0.2985z^2+0.597z+0.2985}$$

$$= \frac{0.2985z^2 + 0.597z + 0.2985}{5.134z^2 - 7.403z + 3.462}$$