UNIT-I

DISCRETE FOURIER TRANSFORM

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LINEARITY

$$If\ x_1(n) \xleftrightarrow{DFT} X_1(k)\ and\ x_2(n) \xleftrightarrow{DFT} X_2(k)\ then,$$

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(k) + a_2 X_2(k)$$

Here a_1 and a_2 are constants

PERIODICITY

x(n+N)=x(n) for all n
X(N+k)=X(k) for all k

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}$$

$$DFT\{x(n+N)\} = \sum_{n=0}^{N-1} x(n+N)\, e^{-j2\pi (n+N)k/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \quad e^{-j2\pi Nk/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \quad e^{-j2\pi k}$$

$$X(K+N) = X(K)$$

TIME SHIFTING

$$If\ x(n) \overset{DFT}{\longleftrightarrow} X(k)\ then,$$

$$x(n-n_0) \overset{DFT}{\longleftrightarrow} e^{-j2\pi k n_0/N} X(k)$$

Proof:

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}$$

$$DFT\{x(n-n_0)\} = \sum_{n=0}^{N-1} x(n-n_0)\, e^{-j2\pi nk/N}$$

Let $n-n_0 = m$; $n = m+n_0$

$$DFT\{x(n-n_0)\} = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi(m+n_0)K/N}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi km/N}\, e^{-j2\pi kn_0/N}$$

$$DFT\{x(n-n_0)\} = e^{-j2\pi kn_0/N}\, X(K)$$

CIRCULAR CONVOLUTION

$$If\ x_1(n) \overset{DFT}{\longleftrightarrow} X_1(k)\ and\ x_2(n) \overset{DFT}{\longleftrightarrow} X_2(k)\ then,$$

$$x_1(n)(N)x_2(n) \overset{DFT}{\longleftrightarrow} X_1(k)X_2(k)$$

Here $x_1(n)(N)x_2(n)$ means circular convolution of $x_1(n)$ and $x_2(n)$. This property states that multiplication of two DFTs is equivalent to circular convolution of their sequences in time domain.

CIRCULAR TIME SHIFT

$$If\ x(n) \overset{DFT}{\longleftrightarrow} X(k)\ then,$$

$$x((n-l))_N \overset{DFT}{\longleftrightarrow} X(k)e^{-j2\pi kl/N}$$

Thus shifting the sequence circularly by ‘l’ samples is equivalent to multiplying its DFT by $e^{-j2\pi kl/N}$

CIRCULAR FREQUENCY SHIFT

$$If\ x(n) \overset{DFT}{\longleftrightarrow} X(k)\ then,$$

$$x(n)e^{j2\pi ln/N} \overset{DFT}{\longleftrightarrow} X((k-l))_N$$

Thus shifting the frequency components of DFT circularly is equivalent to multiplying the time domain sequence by $e^{j2\pi ln/N}$

MULTIPLICATION OF TWO SEQUENCES

$$If\ x_1(n) \overset{DFT}{\longleftrightarrow} X_1(k)\ and\ x_2(n) \overset{DFT}{\longleftrightarrow} X_2(k)\ then,$$

$$x_1(n)x_2(n) \overset{DFT}{\longleftrightarrow} \frac{1}{N} X_1(k)(N)X_2(k)$$

This means multiplication of two sequences in time domain results in circular convolution of their DFTs in frequency domain.

PARSEVAL’S THEOREM

Consider the complex valued sequences x(n) and y(n) then

$$If\ x(n) \overset{DFT}{\longleftrightarrow} X(k)\ and\ \ y(n) \overset{DFT}{\longleftrightarrow} Y(k)\ then$$

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

DFT PROBLEM

Find DFT of x(n) = {1,1,2,2,3,3,} and find the Frequency spectrum

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \quad , \quad K = 0,1 \ldots N-1$$

$$X(0) = \sum_{n=0}^{5} x(n)\, e^{0} = 1+1+2+2+3+3 = 12$$

$$X(1) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi n/6} = \sum_{n=0}^{5} x(n)\, e^{-j\pi n/3}$$

$$= x(0)e^{0} + x(1)\, e^{-j\pi/3} + x(2)\, e^{-j2\pi/3} + x(3)\, e^{-j3\pi/3} + x(4)\, e^{-j4\pi/3} + x(5)\, e^{-j5\pi/3}$$

$$= 1 + 1\,[0.5 - j0.866] + 2\,[-0.5 - j0.866] + 2\,[-1] + 3\,[-0.5 + j0.866] + 3\,[0.5 + j0.866]$$

$$X(1) = -1.5 + j2.598$$

$$X(2) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(2)n/6} = \sum_{n=0}^{5} x(n)\, e^{-j2\pi n/3}$$

$$= 1 + e^{-j2\pi/3} + e^{-j4\pi/3} + e^{-j6\pi/3} + e^{-j8\pi/3} + e^{-j10\pi/3}$$

$$= 1 - 0.5 - j0.866 + 2[-0.5 + j0.866] + 2(1) + 3[-0.5 - j0.866]$$

$$+ 3[-0.5 + j0.866]$$

$$X(2) = -1.5 + j0.866$$

$$X(3) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(3)n/6} = \sum_{n=0}^{5} x(n)\, e^{-j\pi n}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi} + 3e^{-j4\pi} + 3e^{-j5\pi}$$

$$= 1 - 1 + 2 - 2 + 3 - 3 = 0$$

$$X(3) = 0$$

$$X(4) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi 4n/6} = \sum_{n=0}^{5} x(n)\, e^{-j4\pi n/3}$$

$$= 1 + e^{-j4\pi/3} + 2e^{-j8\pi/3} + 2e^{-j12\pi/3} + 3e^{-j16\pi/3} + 3e^{-j20\pi/3}$$

$$= 1 - 0.5 + j0.866 + 2(-0.5 - j0.866) + 2(1) + 3(-0.5 + j0.866)$$

$$+ 3(-0.5 - j0.866)$$

$$X(4) = -1.5 - j0.866$$

$$X(5) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi 5n/6} = \sum_{n=0}^{5} x(n)\, e^{-j5\pi n/3}$$

$$= 1 + e^{-j5\pi/3} + 2e^{-j10\pi/3} + 2e^{-j15\pi/3} + 3e^{-j20\pi/3} + 3e^{-j25\pi/3}$$

$$= 1 + 0.5 + j0.866 + 2(-0.5 + j0.866) + 2(-1) + 3(-0.5 - j0.866)$$

$$+ 3(0.5 - j0.866)$$

$$X(5) = -1.5 - j2.598$$

$$X(k) = \{12, -1.5+j2.598, -1.5+j.866, 0, -1.5-j.866, -1.5-j2.598\}$$

$$|X(k)| = \{12, 2.999, 1.732, 0, 1.732, 2.999\}$$

$$\angle X(k) = \{0, -60, -30, 0, 30, 60\}$$

Find IDFT of X(k) = {3, 2+j, 1, 2-j}

IDFT of X(k) is given by

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{j2\pi nk/N} \qquad , n = 0,1, \ldots N-1$$

Where N=4

Apply n=0, 1, 2, 3 we will get

x(n) = {2, 0, 0, 1}

Fast Fourier Transform (FFT):

FFT is an algorithm that efficiently computes the DFT of a sequence $x(n)$ of length N.

DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1$$

Let W_N be the complex-valued phase factor or twiddle factor, which is an N^{th} root of unity expressed by

$$W_N = e^{-j2\pi/N}$$

The DFT and IDFT using Twiddle factor

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \quad , \quad K = 0, 1 .. N-1$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \quad , \quad n = 0, 1 .. N-1$$

Properties of Twiddle factor

Symmetry: $W_N^{K+\frac{N}{2}} = -W_N^K$

$$W_N^{K+\frac{N}{2}} = W_N^K \; W_N^{N/2}$$

$$= W_N^K \; e^{-j\frac{2\pi}{N}\frac{N}{2}}$$

$$= W_N^K \; e^{-j\pi}$$

$$W_N^{K+\frac{N}{2}} = -W_N^K$$

Periodicity: $W_N^{K+N} = W_N^K$

$$W_N^{K+N} = W_N^K \; W_N^N$$

$$= W_N^K \; e^{-j\frac{2\pi}{N}N}$$

$$W_N^{K+N} = W_N^K$$

Radix 2 FFT Algorithm

By adopting divide and conquer approach, a computationally efficient algorithm for the DFT can be developed. This approach depends on the decomposition of N-point DFT into successfully smaller size DFTs.

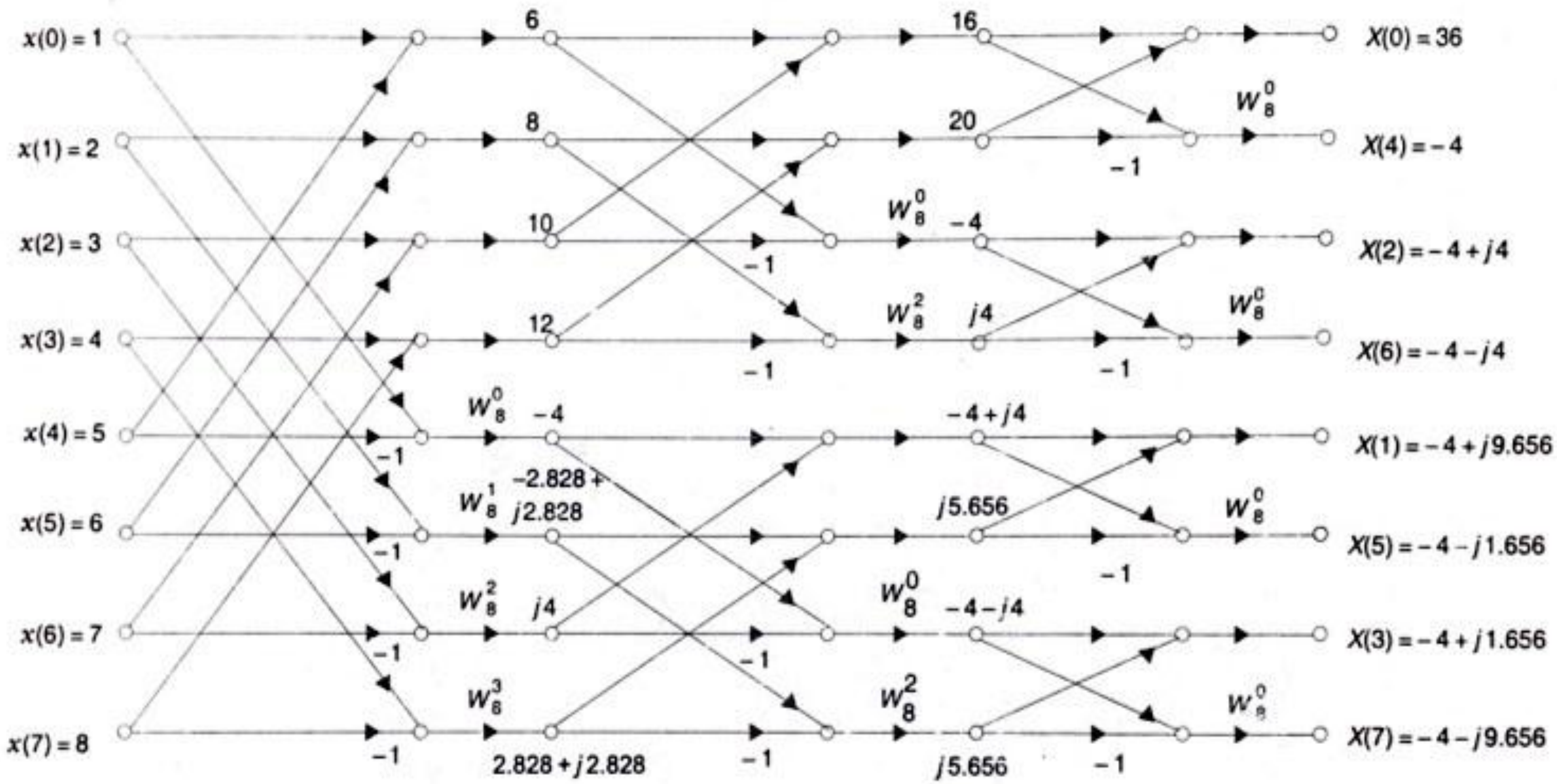
N is factored as $N = r_1 r_2 r_3 \ldots r_L$ where $r_1 = r_2 = r_L = r$ then $N = r^L$

r is called radix of FFT algorithm.

Number of stages $L = \log_2 N$

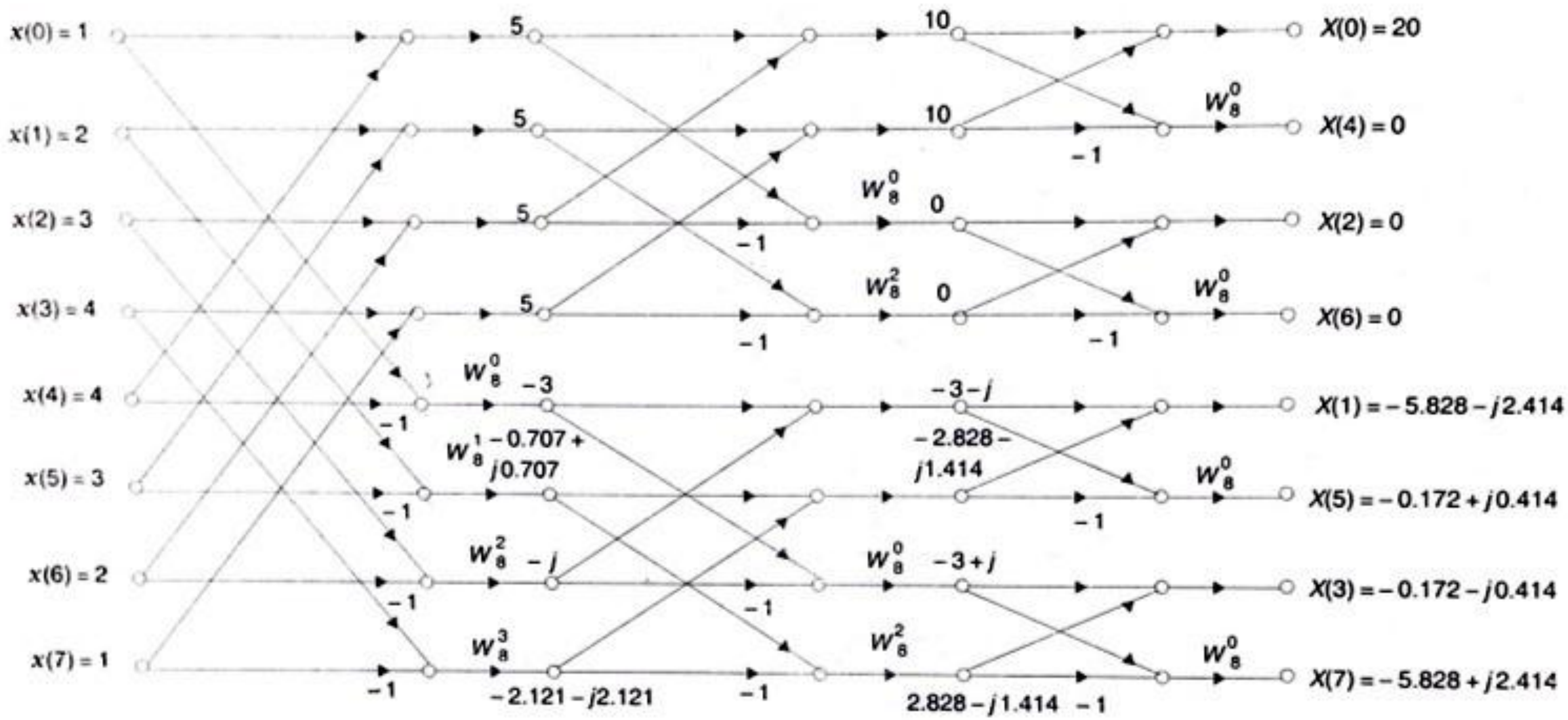
Speed Improvement factor $= \dfrac{N^2}{\frac{N}{2}\log_2 N}$

Find the DFT of x(n) = {1,2,3,4,5,6,7,8} using DIF-FFT algorithm



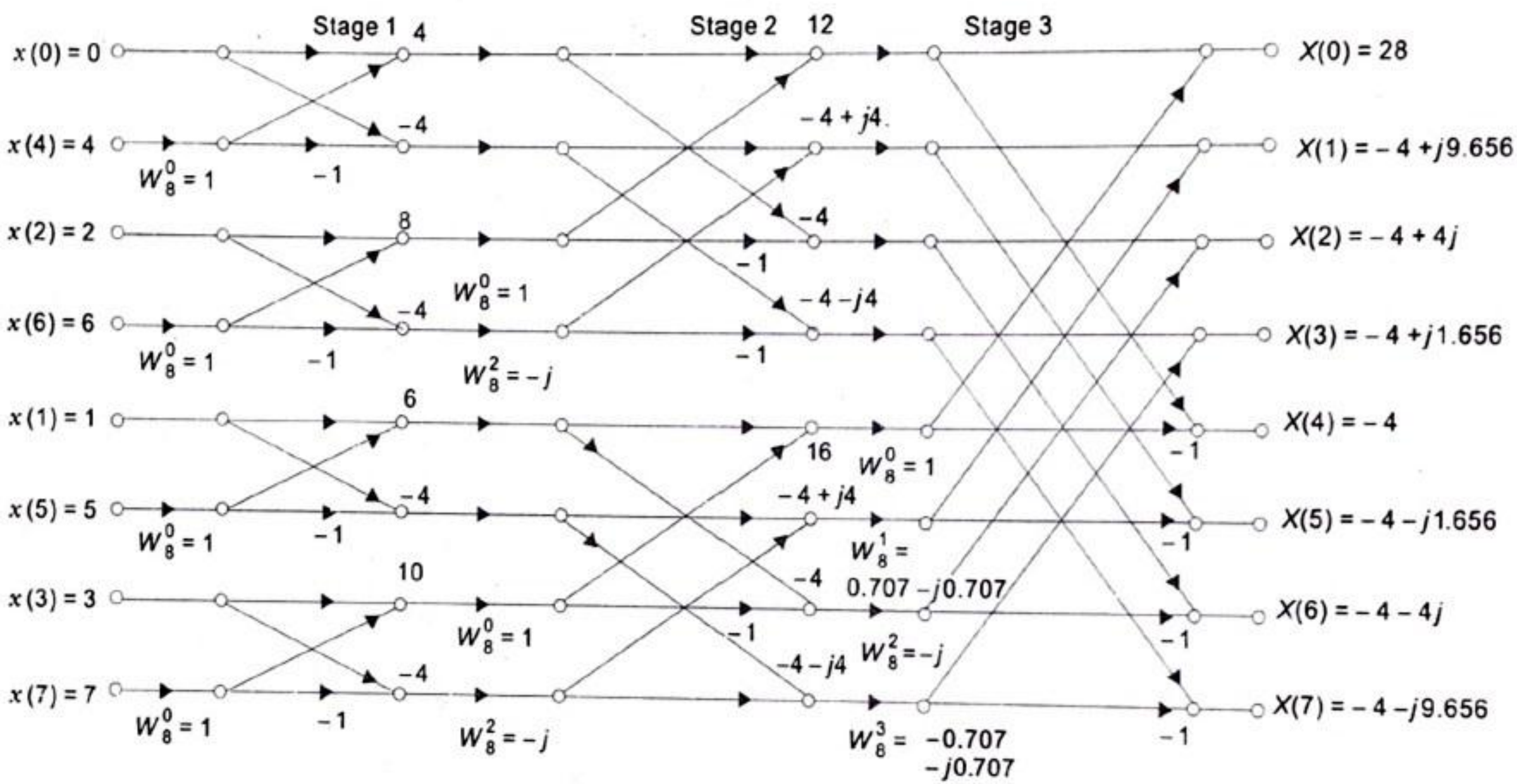
X(k) = {36, -4+j9.656, -4+j4, -4+j1.656, -4, -4-j1.656, -4-j4, -4-j9.656}

Find the DFT of x(n) = {1,2,3,4,4,3,2,1} using DIF-FFT algorithm



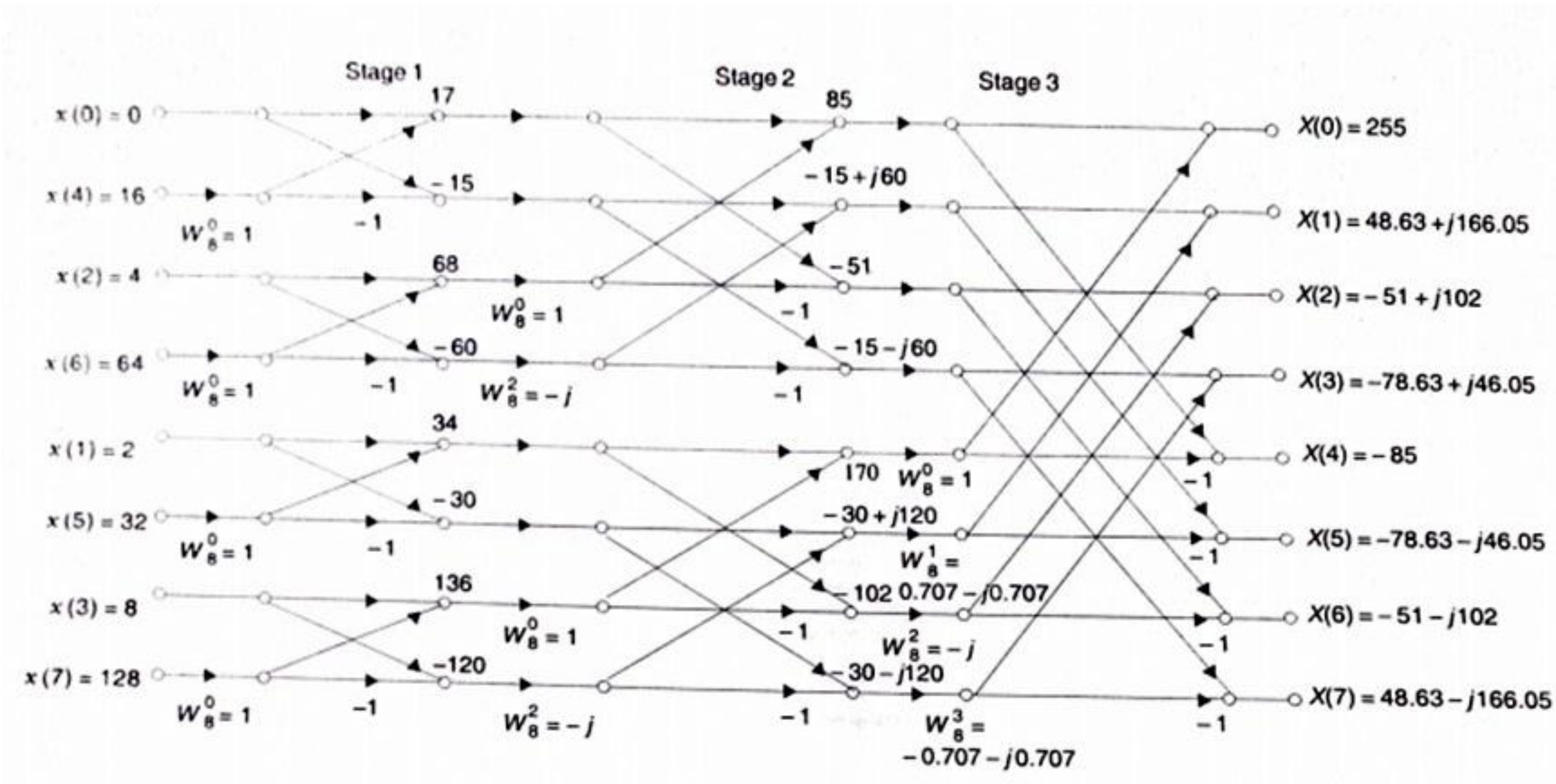
X(k) = {20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0 , -5.828+j2.414}

Find the DFT of x(n) = {0,1,2,3,4,5,6,7} using DIT-FFT algorithm



X(k) = {28, -4+j9.656, -4+j4, -4+j1.656, -4, -4-j1.656, -4-j4, -4-j9.656}

Find the DFT of x(n) = n^2 using DIT-FFT algorithm



X(k) = {256, 48.63+j166.05, -51+j102, -78.63+j46.05, -85, -78.63-j46.05, -51-j102, 48.63-j166.05}