NPR College of Engineering & Technology

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Welcomes you All

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MA8491

NUMERICAL METHODS

OBJECTIVES :

- To introduce the basic concepts of solving algebraic and transcendental equations.
- To introduce the numerical techniques of interpolation in various intervals in real life situations.
- To acquaint the student with understanding of numerical techniques of differentiation and integration which plays an important role in engineering and technology disciplines.
- To acquaint the knowledge of various techniques and methods of solving ordinary differential equations.
- To understand the knowledge of various techniques and methods of solving various types of partial differential equations.

UNIT I SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

Solution of algebraic and transcendental equations - Fixed point iteration method – Newton Raphson method - Solution of linear system of equations - Gauss elimination method – Pivoting - Gauss Jordan method – Iterative methods of Gauss Jacobi and Gauss Seidel - Eigenvalues of a matrix by Power method and Jacobi's method for symmetric matrices.

UNIT II INTERPOLATION AND APPROXIMATION

Interpolation with unequal intervals - Lagrange's interpolation – Newton's divided difference interpolation – Cubic Splines - Difference operators and relations - Interpolation with equal intervals - Newton's forward and backward difference formulae.

UNIT III NUMERICAL DIFFERENTIATION AND INTEGRATION

Approximation of derivatives using interpolation polynomials - Numerical integration using Trapezoidal, Simpson's 1/3 rule – Romberg's Method - Two point and three point Gaussian quadrature formulae – Evaluation of double integrals by Trapezoidal and Simpson's 1/3 rules.

UNIT IV INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Single step methods - Taylor's series method - Euler's method - Modified Euler's method - Fourth order Runge - Kutta method for solving first order equations - Multi step methods - Milne's and Adams - Bash forth predictor corrector methods for solving first order equations.

UNIT V BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Finite difference methods for solving second order two - point linear boundary value problems - Finite difference techniques for the solution of two dimensional Laplace's and Poisson's equations on rectangular domain – One dimensional heat flow equation by explicit and implicit (Crank Nicholson) methods – One dimensional wave equation by explicit method.

TEXTBOOKS :

1. Burden, R.L and Faires, J.D, "Numerical Analysis", 9th Edition, Cengage Learning, 2016.
2. Grewal, B.S., and Grewal, J.S., "Numerical Methods in Engineering and Science", Khanna Publishers, 10th Edition, New Delhi, 2015.

REFERENCES :

1. Brian Bradie, "A Friendly Introduction to Numerical Analysis", Pearson Education, Asia, New Delhi, 2007.
2. Gerald. C. F. and Wheatley. P. O., "Applied Numerical Analysis", Pearson Education, Asia, 6th Edition, New Delhi, 2006.
3. Mathews, J.H. "Numerical Methods for Mathematics, Science and Engineering", 2nd Edition, Prentice Hall, 1992.
4. Sankara Rao. K., "Numerical Methods for Scientists and Engineers", Prentice Hall of India Pvt. Ltd, 3rd Edition, New Delhi, 2007.
5. Sastry, S.S, "Introductory Methods of Numerical Analysis", PHI Learning Pvt. Ltd, 5th Edition, 2015.

UNIT – 3

NUMERICAL DIFFERENTIATION AND INTEGRATION

Newton's forward formula for derivatives

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \cdots \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y''' = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(24u-36)}{4!}\Delta^4 y_0 + \cdots\right]$$

(1) Find the first three derivatives of $f(x)$ at $x = 1.5$ & at $x = 4.0$ using Newton's forward interpolation formula to the data given below.

x	1.5	2	2.5	3	3.5	4
y	3.375	7	13.625	24	38.875	59

Soln

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(24u-36)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$u = \frac{x-x_0}{h} = \frac{x-1.5}{0.5}$$

When $x = 1.5$ $\boxed{u=0}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2	7		3			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4	59					

$$f'(1.5) = \frac{1}{0.5}\left[3.625 + (0-1)\cdot\frac{3}{2} + \frac{2}{6}(0.75)\right]$$

$$= \frac{1}{0.5}\left[3.625 - 1.5 + 0.25\right]$$

$$= 4.75$$

$$f''(1.5) = \frac{1}{0.5^2}\left[3 + (-6)\times\frac{0.75}{6}\right]$$

$$= \frac{1}{0.5^2}\left[3 - 0.75\right] = 9$$

$$f'''(1.5) = \frac{1}{0.5^3}\left[0.75\right] = 6$$

Newton's Backward Interpolation formula

$$y' = \frac{1}{h}\left[\nabla y_n + \frac{(2v+1)}{2!}\nabla^2 y_n + \frac{(3v^2+6v+2)}{3!}\nabla^3 y_n + \frac{(4v^3+18v^2+22v+6)}{4!}\nabla^4 y_n + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\nabla^2 y_n + (6v+6)\frac{\nabla^3 y_n}{3!} + \frac{(12v^2+36v+22)}{4!}\nabla^4 y_n + \cdots\right]$$

$$y''' = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{(24v+36)}{4!}\nabla^4 y_n + \cdots\right]$$

$$v = \frac{x - x_n}{h} = \frac{x-4}{0.5}$$

When $x = 4 \Rightarrow \boxed{v = 0}$

$$y' = \frac{1}{0.5}\left[20.125 + \frac{1}{2}\times 5.25 + \frac{2}{6}\times 0.75\right]$$
$$= 46$$
$$y'' = \frac{1}{0.5^2}\left[5.25 + 6\times\frac{0.75}{6}\right] = 24$$
$$y''' = \frac{1}{0.5^3}\left[0.75\right] = 6.$$

(2) For the given data, find the first two derivatives at $x = 1.1$

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Soln

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$u = \frac{x-x_0}{h} = \frac{x-1.0}{0.1}$$

At $x = 1.1$ $\quad u = \frac{1.1-1.0}{0.1} = 1.$

$y' = \frac{1}{0.1}[0.4$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	7.989					
		0.4140				
1.1	8.403		−0.0360			
		0.3780		0.0060		
1.2	8.781		−0.03		−0.0020	
		0.3480		0.0040		0.001
1.3	9.129		−0.0260		−0.0010	
		0.3220		0.003		0.003
1.4	9.451		−0.0230		0.002	
		0.2990		0.0050		
1.5	9.750		−0.0180			Δ^6
		0.2810				
1.6	10.031					0.00

$$y'(1.1) = \frac{1}{0.1}\left[0.414 + \frac{(2-1)}{2}(-0.0360) + \frac{(3-6+2)}{6}(0.006) + \frac{(4-18+22-6)}{24}(-0.002)\right]$$

$$= \frac{1}{0.1}\left[0.414 - 0.0180 - 0.0010 - 0.0002\right]$$

$$= 3.9480$$

$$y''(1.1) = \frac{1}{(0.1)^2}\left[(-0.0360) + \frac{(6-6)}{6}(0.0060)\right] + \left(\frac{12-36+22}{24}\right)(-0.0020)\bigg]$$

$$= 100\left[-0.0360 + 0\right] + \frac{(-2)}{24}(-0.0020)\bigg]$$

$= -36 + 0.00016$

$= \cancel{-35.9998}\ -3.584$

① find the first derivatives of $f(x)$ at $x=2$ for the data $f(-1) = -21$, $f(1) = 15$, $f(2) = 12$, $f(3) = 3$. using Newton's divided difference formula.

Soln

x	-1	1	2	3
y	-21	15	12	3

The Newton's divided difference formula is

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \cdots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

$$y = -21 + (x+1)\,18 + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)\,(1)$$

$$= -21 + 18x + 18 - 7(x^2-1) + (x^2-1)(x-2)$$

$$= -21 + 18x + 18 - 7x^2 + 7 + x^3 - 2x^2 - x + 2$$

$$y = x^3 - 9x^2 + 17x + 6$$

$$y' = 3x^2 - 18x + 17$$

$$y'(2) = -7$$

Numerical Integration

Trapezoidal rule

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[(\text{sum of first and last ordinate}) + 2(\text{sum of remaining ordinates}) \right]$$

$$h = \frac{b-a}{n}$$

Simpsons 1/3 rule

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\left[(\text{first} + \text{Last}) + 4(\text{sum of odd ordinates}) + 2(\text{sum of even ordinates}) \right]$$

$$h = \frac{b-a}{n} - [\text{multiples of } 2]$$

Simpsons 3/8 rule

$$I = \frac{3h}{8}\left[(\text{first Last}) + 2(\text{sum of multiples of } 3) + 3(\text{sum of non-multiples of } 3) \right]$$

$$h = \frac{b-a}{n} \quad [\text{multiples of } 3]$$

(3) Evaluate $I = \int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method. Hence deduce an approximate value of π.

Soln

$a = 0 \; ; \; b = 1$

$f(x) = \frac{1}{1+x^2}$

I $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

x	0	0.5	1
$f(x)$	1	0.8	0.5

$$I_1 = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1)\right]$$

$$= \frac{0.5}{2}\left[(1 + 0.5) + 2 \times 0.8\right]$$

$$I_1 = 0.7750$$

II $h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9412	0.8	0.64	0.5

$$I_2 = \frac{0.25}{2}\left[(1+0.5)+2(0.9412+0.8+0.64)\right]$$

$$\boxed{I_2 = 0.7828}$$

III $\quad h = \frac{b-a}{8} = \frac{1-0}{8} = 0.125$

x	0	0.12	0.25	0.375	0.5
$f(x)$	1	0.9846	0.9412	0.8767	0.8

0.625	0.75	0.875	1
0.7191	0.64	0.5664	0.5

$$I_3 = \frac{0.5}{2}\left[(1+0.5)+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)\right]$$

$$\boxed{I_3 = 0.7848}$$

Romberg for I_1, I_2

$$I_4 = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7854$$

Romberg for I_2, I_3

$$I_5 = I_3 + \left(\frac{I_3 - I_2}{3} \right) = 0.7855$$

Romberg for I_4, I_5

$$I = I_5 + \left(\frac{I_5 - I_4}{3} \right) = 0.7855$$

$$I = \int_0^1 \frac{dx}{1+x^2}$$

$$0.7855 = \left[\tan^{-1} x \right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$\frac{\pi}{4} = 0.7855$$

$$\pi = 3.1420.$$

(3) Using Romberg Integration, evaluate

$$\int_0^1 \frac{dx}{1+x}$$

Soln

I) Here $a=0,\ b=1$

x	0	0.5	1
$\frac{1}{1+x}$	1	0.6667	0.5

$$I_1 = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1)\right]$$

$$= \frac{0.5}{2}\left[(1+0.5) + 2(0.6667)\right]$$

$$\boxed{I_1 = 0.7084}$$

II) $h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.8	0.6667	0.5714	0.5

$$I_2 = \frac{0.25}{2}\left[(1+0.5) + 2(0.8 + 0.6667 + 0.5714)\right]$$

$$\boxed{I_2 = 0.6970}$$

III $h = \frac{b-a}{8} = \frac{1-0}{8} = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

$$I_3 = \frac{0.125}{2}\left[(1+0.5) + 2(0.8889+0.8+0.7273+0.6667+0.6154+0.5714+0.5333)\right]$$

$$\boxed{I_3 = 0.6941.}$$

Romberg for I_1, I_2

$$I_4 = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.6970 + \left(\frac{0.6970 - 0.7084}{3}\right)$$

$$\boxed{I_4 = 0.6932}$$

Romberg for I_2, I_3

$$I_5 = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$= 0.6941 + \left(\frac{0.6941 - 0.6970}{3}\right).$$

$$\boxed{I_5 = 0.6931}$$

Romberg for I_4, I_5

$$I_6 = I_5 + \left(\frac{I_5 - I_4}{3}\right)$$

$$\boxed{I_6 = 0.6931}$$

Gauss Quadrature formula

Quadrature:

The Process of finding a definite integral from a tabulated values of a function is known as Quadrature.

Gaussian two point Quadrature formula

Let $I = \int_a^b f(x)\, dx$.

Take $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$

$dx = \left(\frac{b-a}{2}\right) dt$.

By using this transformation

$$I = \int_{-1}^{1} g(t)\,dt = g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

① Evaluate $\int_{-1}^{1} e^{-x^2} \cos x \, dx$ by Gauss two Point Quadrature formula.

Soln

$$I = \int_{-1}^{1} e^{-x^2} \cos x \, dx$$

$$f(x) = e^{-x^2} \cos x$$

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= e^{-\left(\frac{1}{\sqrt{3}}\right)^2} \cos\left(\frac{-1}{\sqrt{3}}\right) + e^{-\left(\frac{1}{\sqrt{3}}\right)^2} \cos\left(\frac{1}{\sqrt{3}}\right)$$

$$= e^{-1/3} \cos\left(\frac{-1}{\sqrt{3}}\right) + e^{-1/3} \cos\left(\frac{1}{\sqrt{3}}\right)$$

$$= e^{-1/3} \left[\cos\left(-1/\sqrt{3}\right) + \cos\left(1/\sqrt{3}\right) \right]$$

$$I = 1.2008$$

② Apply Gauss two point formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2} dx$

Soln

$$I = \int_{-1}^{1} \frac{1}{1+x^2} dx$$

$= \frac{\pi}{4}[0.3259 + 0.9454]$

$= 0.9985$

Gaussian Three Point Quadrature formula:

$$I = \int_a^b f(x)\,dx$$

Take $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$

$$dx = \left(\frac{b-a}{2}\right)dt$$

$$I = \int_{-1}^{1} g(t)\,dt = \frac{5}{9}\left[g\left(-\sqrt{\tfrac{3}{5}}\right) + g\left(\sqrt{\tfrac{3}{5}}\right)\right] + \frac{8}{9}g(0)$$

① Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using 3 point Quadrature formula.

Soln

$$I = \int_0^1 \frac{dx}{1+x^2}$$

$f(x) = \frac{1}{1+x^2}$, $a = 0$, $b = 1$

Take $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$

$$dx = \left(\frac{b-a}{2}\right) dt$$

$$\Rightarrow x = \frac{1}{2} + \frac{1}{2}t$$

$$dx = \frac{1}{2}\,dt$$

$$I = \int_{-1}^{1} \frac{\frac{1}{2}\,dt}{1+\left(\frac{1+t}{2}\right)^2} = \frac{1}{2}\int_{-1}^{1} \frac{dt}{1+\left(\frac{1+t}{2}\right)^2}$$

$$\therefore \quad g(t) = \frac{1}{1+\left(\frac{1+t}{2}\right)^2}$$

$$I = \frac{1}{2}\left[\frac{5}{9}\left[g\left(-\sqrt{\tfrac{3}{5}}\right) + g\left(\sqrt{\tfrac{3}{5}}\right)\right] + \frac{8}{9}g(0)\right]$$

$$= \frac{1}{2}\left[\frac{5}{9}\left(\frac{1}{1+\left(\frac{1+(-\sqrt{3/5})}{2}\right)^2} + \frac{1}{1+\left(\frac{1+(\sqrt{\frac{3}{5}})}{2}\right)^2}\right) + \frac{8}{9}\left[\frac{1}{1+(1/2)^2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{5}{9}\left(0.9875 + 0.5595\right) + 0.7111\right]$$

$$= 0.7853.$$

Double Integration

Trapezoidal rule:

$$I = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

$$I = \frac{hk}{4}\Big[\text{Sum of four corners} + 2(\text{Sum of remaining boundary values}) + 4(\text{Sum of interior Values})\Big]$$

Simpson's rule

$$I = \frac{hk}{9}\Big[\text{Sum of four corners} + \underbrace{2(\text{Sum of odd position values}) + 4(\text{Sum of even position values})}_{\text{Boundary}}$$

$$+ \underbrace{4(\text{Sum of odd position values}) + 8(\text{Sum of even position values})}_{\text{odd rows}}$$

$$+ \underbrace{8(\text{Sum of odd position values}) + 16(\text{Sum of even position values})}_{\text{even rows}}\Big]$$

$$\cancel{I} = \frac{hk}{4}\Big[\cancel{\text{Sum of four corn}}$$

(2) Evaluate $\int_1^2 \int_1^2 \frac{1}{x^2+y^2}\, dx\, dy$, numerically using $h = 0.2$, along x-direction and $k = 0.25$ along y-direction

Soln

$$I = \int_1^2 \int_1^2 \frac{1}{x^2+y^2}\, dx\, dy$$

$$f(x, y) = \frac{1}{x^2+y^2}$$

By Trapezoidal

$$I = \frac{hk}{4}\Big[\text{Sum of four corners} + 2(\text{Sum of remaining boundary}) + 4(\text{Sum of interiors})\Big]$$

y \ x	1	1.2	1.4	1.6	1.8	2
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.5	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

$$I = \frac{(0.2)(0.25)}{4}\Big[0.5+0.2+0.2+0.125$$
$$+2(0.4098+0.3378+0.2809+0.2359$$
$$+0.1798+0.16+0.1416$$
$$+0.1381+0.1524+0.1679+0.1838$$
$$+0.2462+0.2710+0.3331)$$
$$+4(0.3331+0.2839+0.2426$$
$$+0.2082+0.2710+0.2375$$
$$+0.2079+0.1821+0.2221$$
$$+0.1991+0.1779+0.1587)\Big]$$

$$= \frac{(0.2)(0.25)}{4}\left[1.025+6.6642+10.8964)\right]$$

$$= 0.2323.$$