DIMENSIONAL ANALYSIS

(OPEN CHANNEL FLOW AND HYDRAULIC MACHINERY)

UNIT – II

Learning Objectives

1. Introduction to Dimensions & Units
2. Use of Dimensional Analysis
3. Dimensional Homogeneity
4. Methods of Dimensional Analysis
5. Rayleigh's Method

Learning Objectives

6. Buckingham's Method
7. Model Analysis
8. Similitude
9. Model Laws or Similarity Laws
10. Model and Prototype Relations

Introduction

- Many practical real flow problems in fluid mechanics can be solved by using equations and analytical procedures. However, solutions of some real flow problems depend heavily on experimental data.

- Sometimes, the experimental work in the laboratory is not only time-consuming, but also expensive. So, the main goal is to extract maximum information from fewest experiments.

- In this regard, dimensional analysis is an important tool that helps in correlating analytical results with experimental data and to predict the prototype behavior from the measurements on the model.

Dimensions and Units

In dimensional analysis we are only concerned with the nature of the dimension i.e. its quality <u>not</u> its quantity.

- Dimensions are properties which can be measured.
  Ex.: Mass, Length, Time etc.,
- Units are the standard elements we use to quantify these dimensions.
  Ex.: Kg, Metre, Seconds etc.,

The following are the Fundamental Dimensions (MLT)

Mass	kg	M
Length	m	L
Time	s	T

Secondary or Derived Dimensions

Secondary dimensions are those quantities which posses more than one fundamental dimensions.

1. Geometric

a)	Area	m^2		L^2
b)	Volume	m^3		L^3

2. Kinematic

a)	Velocity	m/s	L/T	$L.T^{-1}$
b)	Acceleration	m/s^2	L/T^2	$L.T^{-2}$

3. Dynamic

a)	Force	N	ML/T	$M.L.T^{-1}$
b)	Density	kg/m^3	M/L^3	$M.L^{-3}$

Problems

Find Dimensions for the following:

1. Stress / Pressure
2. Work
3. Power
4. Kinetic Energy
5. Dynamic Viscosity
6. Kinematic Viscosity
7. Surface Tension
8. Angular Velocity
9. Momentum
10. Torque

Use of Dimensional Analysis

1. Conversion from one dimensional unit to another
2. Checking units of equations (Dimensional Homogeneity)
3. Defining dimensionless relationship using
   a) Rayleigh's Method
   b) Buckingham's π-Theorem
4. Model Analysis

Dimensional Homogeneity

Dimensional Homogeneity means the dimensions in each equation on both sides equal.

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S. $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S. $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S. $=$ Dimension of R.H.S. $= LT^{-1}$

$\therefore$ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

Problems

Check Dimensional Homogeneity of the following:

1. $Q = AV$
2. $E_K = v^2/2g$

Rayeligh's Method

To define relationship among variables

This method is used for determining the expression for a variable which depends upon maximum three or four variables only.

Rayeligh's Method

Methodology:

Let X is a function of X_1 , X_2, X_3 and mathematically it can be written as

$X = f(X_1, X_2, X_3)$

This can be also written as

$X = K\ (X_1^a\ , X_2^b\ , X_3^c\ )$ where K is constant and a, b and c are arbitrarily powers

The values of a, b and c are obtained by comparing the powers of the fundamental dimension on both sides.

Rayeligh's Method

Problem: Find the expression for Discharge Q in a open channel flow when Q is depends on Area A and Velocity V.

Solution:

$Q = K.A^a.V^b$ → 1

where K is a Non-dimensional constant

Substitute the dimensions on both sides of equation 1

$M^0 L^3 T^{-1} = K. (L^2)^a.(LT^{-1})^b$

Equating powers of M, L, T on both sides,

Power of T, $-1 = -b$ → $b=1$

Power of L, $3= 2a+b$ → $2a = 2-b = 2-1 = 1$

Substituting values of a, b, and c in Equation 1m

$Q = K. A^1. V^1 = V.A$

Problem : Find the equation for the power developed by a pump if it depends on head H discharge Q and specific weight γ of the fluid.

Solution:

Power	$= L^2MT^{-3}$
Head	$= LM^oT^o$
Discharge	$= L^3M^oT^{-1}$
Specific Weight	$= L^{-2}MT^{-2}$

$$P = f(H, Q, \gamma)$$

$$P = K \cdot H^a \cdot Q^b \cdot \gamma^c$$

$$[P] = [H]^a \cdot [Q]^b \cdot [\gamma]^c$$

$$[L^2MT^{-3}] = [LM^oT^o]^a \cdot [L^3M^oT^{-1}]^b \cdot [L^{-2}MT^{-2}]^c$$

Equating the powers of M, L and T on both sides,

Power of M, $1 = c$

Power of T, $-3 = -b - 2$ or $b = -2 + 3$ or $\mathbf{b = 1}$

Power of L, $2 = a + 3b - 2c$ or $2 = a + 3 - 2$ or $\mathbf{a = 1}$

Substituting the values of a, b and c

$$P = K \cdot H^1 \cdot Q^1 \cdot \gamma^1$$

$P = K \cdot H \cdot Q \cdot \gamma$ When, $K = 1$ $\mathbf{P = H \cdot Q \cdot \gamma}$

Problem 3: Find an expression for drag force R on a smooth sphere of diameter D moving with uniform velocity V in a fluid of density ρ and dynamic viscosity μ..

Solution:

Force	$= LMT^{-2}$
Diameter	$= LM^oT^o$
Velocity	$= LM^oT^{-1}$
Mass density	$= L^3MT^o$

$$R = f(D, V, \rho, \mu)$$

$$R = K \cdot D^a \cdot V^b \cdot \rho^c, \mu^d$$

$$[R] = [D]^a \cdot [V]^b \cdot [\rho]^c \cdot [\mu]^d$$

$$[LMT^{-2}] = [LM^oT^o]^a \cdot [LM^oT^{-1}]^b \cdot [L^{-3}MT^o]^c \cdot [L^{-1}MT^{-1}]^d$$

Equating the powers of M, L and T on both sides,

Power of M, $1 = c + d$ or $\mathbf{c = 1 - d}$

Power of T, $-2 = -b - d$ or $\mathbf{b = 2 - d}$

Power of L, $1 = a + b - 3c - d$ or $1 = a + 2 - d - 3(1 - d) - d$

$1 = a + 2 - d - 3 + 3d - d$ or $\mathbf{a = 2 - d}$

Substituting teh values of a, b, and c

$$R = K \cdot D^{2-d} \cdot V^{2-d} \cdot \rho^{1-d}, \mu^d = K\frac{D^2}{D^d} \cdot \frac{V^2}{V^d} \cdot \frac{\rho}{\rho^d} \cdot \mu^d$$

$$= K \cdot \rho V^2 D^2 \left[\frac{\mu}{\rho VD}\right]^d = \rho V^2 D^2 \phi \left[\frac{\mu}{\rho VD}\right] = \rho V^2 D^2 \phi \left[\frac{\rho VD}{\mu}\right]$$

Buckingham's π-Theorem

This method of analysis is used when number of variables are more.

Theorem:

If there are n variables in a physical phenomenon and those n variables contain m dimensions, then variables can be arranged into (n-m) dimensionless groups called Φ terms.

Explanation:

If $f(X_1, X_2, X_3, \ldots\ldots\ldots X_n) = 0$ and variables can be expressed using m dimensions then
$f(\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots \pi_{n-m}) = 0$ where, $\pi_1, \pi_2, \pi_3, \ldots$ are dimensionless groups.
Each π term contains (m + 1) variables out of which m are of repeating type and one is of non-repeating type.
Each π term being dimensionless, the dimensional homogeneity can be used to get each π term.

π denotes a non-dimensional parameter

Buckingham's π-Theorem

Selecting Repeating Variables:

1. Avoid taking the quantity required as the repeating variable.
2. Repeating variables put together should not form dimensionless group.
3. No two repeating variables should have same dimensions.
4. Repeating variables can be selected from each of the following properties.
   - Geometric property → Length, height, width, area
   - Flow property → Velocity, Acceleration, Discharge
   - Fluid property → Mass density, Viscosity, Surface tension

Problem 12.11 *The pressure difference Δp in a pipe of diameter D and length l due to viscous flow depends on the velocity V, viscosity μ and density ρ. Using Buckingham's π-theorem, obtain an expression for Δp.*

Solution.

Δp is a function of D, l, V, μ, ρ or $\Delta p = f(D, l, V, \mu, \rho)$

or $\quad f_1(\Delta p, D, l, V, \mu, \rho) = 0 \quad$ ...(*i*)

Total number of variables, $\quad n = 6$

Number of fundamental dimension, $m = 3$

Number of π-terms $\quad = n - 3 = 6 - 3 = 3$

Hence equation (*i*) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0 \quad$ ...(*ii*)

Each π-term contains $m + 1$ variables, *i.e.*, $3 + 1 = 4$ variable. Out of four variables, three are repeating variables.

Choosing D, V, μ as repeating variables, we have π-terms as

$$\pi_1 = D^{a_1} . V^{b_1} . \mu^{c_1} . \Delta p$$

$$\pi_2 = D^{a_2} . V^{b_2} . \mu^{c_2} . l$$

$$\pi_3 = D^{a_3} . V^{b_3} . \mu^{c_3} . \rho$$

First π-term $\pi_1 = D^{a_1} . V^{b_1} . \mu^{c_1} . \Delta p$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-1}T^{-1})^{c_1} . ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_1 + 1$, $\therefore \; c_1 = -1$

Power of L, $0 = a_1 + b_1 - c_1 - 1$, $\therefore \; a_1 = -b_1 + c_1 + 1 = 1 - 1 + 1 = 1$

Power of T, $0 = -b_1 - c_1 - 2$, $\therefore \; b_1 = -c_1 - 2 = 1 - 2 = -1$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^1 . V^{-1} . \mu^{-1} . \Delta p = \frac{D\Delta p}{\mu V}.$$

Second π-term $\pi_2 = D^{a_2} . V^{b_2} . \mu^{c_2} . l$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-1}T^{-1})^{c_2} . L.$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_2$, $\therefore \; c_2 = 0$

Power of L, $0 = a_2 + b_2 - c_2 + 1$, $\therefore \; a_2 = -b_2 + c_2 - 1 = -1$

Power of T, $0 = -b_2 - c_2$, $\therefore \; b_2 = -c_2 = 0$

Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-1} . V^0 . \mu^0 . l = \frac{l}{D}.$$

Third π-term $\pi_3 = D^{a_3} . V^{b_3} . \mu^{c_3} . \rho$

Substituting the dimension on both sides,

$$M^0L^0T^0 = L^{a_3} . (LT^{-1})^{b_3} . (ML^{-1}T^{-1})^{c_3} . ML^{-3}.$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_3 + 1,$ $\therefore$ $c_3 = -1$

Power of L, $0 = a_3 + b_3 - c_3 - 3,$ $\therefore$ $a_3 = -b_3 + c_3 + 3 = -1 - 1 + 3 = 1$

Power of T, $0 = -b_3 - c_3,$ $\therefore$ $b_3 = -c_3 = -(-1) = 1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^1 . V^1 . \mu^{-1} . \rho = \frac{\rho DV}{\mu}.$$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*),

$$f_1\left(\frac{D\Delta p}{\mu V}, \frac{l}{D}, \frac{\rho DV}{\mu}\right) = 0 \quad \text{or} \quad \frac{D\Delta p}{\mu V} = \phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right] \quad \text{or} \quad \Delta p = \frac{\mu V}{D}\phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right]$$

Experiments show that the pressure difference Δp is a linear function $\frac{l}{D}$. Hence $\frac{l}{D}$ can be taken out of the functional as

$$\Delta p = \frac{\mu V}{D} \times \frac{L}{D}\phi\left[\frac{\rho DV}{\mu}\right]. \textbf{ Ans.}$$

Expression for difference of pressure head for viscous flow

$$h_f = \frac{\Delta p}{\rho g} = \frac{\mu V}{D} \times \frac{l}{D} \times \frac{1}{\rho g}\phi[R_e] \qquad \left\{\because \frac{\rho DV}{\mu} = R_e\right\}$$

$$= \frac{\mu VL}{wD^2}\phi[R_e]. \textbf{ Ans.}$$

Model Analysis

For predicting the performance of the hydraulic structures (such as dams, spillways etc.) or hydraulic machines (such as turbines, pumps etc.) before actually constructing or manufacturing, models of the structures or machines are made and tests are conducted on them to obtain the desired information.

Model is a small replica of the actual structure or machine

The actual structure or machine is called as **Prototype**

Models can be smaller or larger than the Prototype

Model Analysis is actually an experimental method of finding solutions of complex flow problems.

Similitude or Similarities

Similitude is defined as the similarity between the model and prototype in every aspect, which means that the model and prototype have similar properties.

Types of Similarities:

1. Geometric Similarity → Length, Breadth, Depth, Diameter, Area, Volume etc.,
2. Kinematic Similarity → Velocity, Acceleration etc.,
3. Dynamic Similarity → Time, Discharge, Force, Pressure Intensity, Torque, Power

Geometric Similarity

The geometric similarity is said to be exist between the model and prototype if the ratio of all corresponding linear dimensions in the model and prototype are equal.

$$\frac{L_P}{L_m} = \frac{B_P}{B_m} = \frac{D_P}{D_m} = L_r$$

$$\frac{A_P}{A_m} = L_r^2$$

$$\frac{V_P}{V_m} = L_r^3$$

where L_r is Scale Ratio

Kinematic Similarity

The kinematic similarity is said exist between model and prototype if the ratios of velocity and acceleration at corresponding points in the model and at the corresponding points in the prototype are the same.

$$\frac{V_P}{V_m} = V_r$$

$$\frac{a_P}{a_m} = a_r$$

where V_r is Velocity Ratio

where a_r is Acceleration Ratio

Also the directions of the velocities in the model and prototype should be same

Dynamic Similarity

The dynamic similarity is said exist between model and prototype if the ratios of corresponding forces acting at the corresponding points are equal

$$\frac{F_P}{F_m} = F_r$$

where F_r is Force Ratio

It means for dynamic similarity between the model and prototype, the dimensionless numbers should be same for model and prototype.

Types of Forces Acting on Moving Fluid

1. Inertia Force, F_i

- It is the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration.
- It always exists in the fluid flow problems

Types of Forces Acting on Moving Fluid

1. Inertia Force, F_i
2. Viscous Force, F_v

- It is equal to the product of shear stress due to viscosity and surface area of the flow.

Types of Forces Acting on Moving Fluid

1. Inertia Force, F_i
2. Viscous Force, F_v
3. Gravity Force, F_g

- It is equal to the product of mass and acceleration due to gravity of the flowing fluid.

Types of Forces Acting on Moving Fluid

1. Inertia Force, F_i
2. Viscous Force, F_v
3. Gravity Force, F_g
4. Pressure Force, F_p

➢ It is equal to the product of pressure intensity and cross sectional area of flowing fluid

Types of Forces Acting on Moving Fluid

1. Inertia Force, F_i
2. Viscous Force, F_v
3. Gravity Force, F_g
4. Pressure Force, F_p
5. Surface Tension Force, F_s

- It is equal to the product of surface tension and length of surface of the flowing

Types of Forces Acting on Moving Fluid

1. Inertia Force, F_i
2. Viscous Force, F_v
3. Gravity Force, F_g
4. Pressure Force, F_p
5. Surface Tension Force, F_s
6. Elastic Force, F_e

- It is equal to the product of elastic stress and area of the flowing fluid

Dimensionless Numbers

Dimensionless numbers are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force.

1. Reynold's number, R_e = $\frac{\text{Inertia Force}}{\text{Viscous Force}} = \frac{\rho VL}{\mu} \, or \, \frac{\rho VD}{\mu}$
2. Froude's number, F_e = $\sqrt{\frac{\text{Inertia Force}}{\text{Gravity Force}}} = \frac{V}{\sqrt{Lg}}$
3. Euler's number, E_u = $\sqrt{\frac{\text{Inertia Force}}{\text{Pressure Force}}} = \frac{V}{\sqrt{p/\rho}}$
4. Weber's number, W_e = $\sqrt{\frac{\text{Inertia Force}}{\text{Surface Tension Force}}} = \frac{V}{\sqrt{\sigma/\rho L}}$
5. Mach's number, M = $\sqrt{\frac{\text{Inertia Force}}{\text{Elastic Force}}} = \frac{V}{C}$

12.8.1 Reynold's Number (R_e).

It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for Reynold's number is obtained as

$$\text{Inertia force } (F_i) = \text{Mass} \times \text{Acceleration of flowing fluid}$$

$$= \rho \times \text{Volume} \times \frac{\text{Velocity}}{\text{Time}} = \rho L^3 \frac{V}{T} = \rho L^2 \frac{L}{T} V = \rho L^2 V^2$$

$$= \rho A V^2 \qquad \text{...(12.11)}$$

$$\text{Viscous force } (F_v) = \text{Shear stress} \times \text{Area} \qquad \left\{\because \tau = \mu \frac{du}{dy} \quad \therefore \text{Force} = \tau \times \text{Area}\right\}$$

$$= \tau \times A$$

$$= \left(\mu \frac{du}{dy}\right) \times A = \mu \cdot \frac{V}{L} \times A \qquad \left\{\because \frac{du}{dy} = \frac{V}{L}\right\}$$

By definition, Reynold's number,

$$R_e = \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \cdot \frac{V}{L} \times A} = \frac{\rho V L}{\mu}$$

$$= \frac{V \times L}{(\mu / \rho)} = \frac{V \times L}{\nu} \qquad \left\{\because \frac{\mu}{\rho} = \nu = \text{Kinematic viscosity}\right\}$$

In case of pipe flow, the linear dimension L is taken as diameter, d. Hence Reynold's number for pipe flow,

$$R_e = \frac{V \times d}{\nu} \quad \text{or} \quad \frac{\rho V d}{\mu}. \qquad \text{...(12.12)}$$

12.8.2 Froude's Number (F_e).

The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as

$$F_e = \sqrt{\frac{F_i}{F_g}}$$

where F_i from equation (12.11) $= \rho A V^2$

and F_g = Force due to gravity

$= \text{Mass} \times \text{Acceleration due to gravity}$

$= \rho \times \text{Volume} \times g = \rho \times L^3 \times g$ $\{\because \ \text{Volume} = L^3\}$

$= \rho \times L^2 \times L \times g = \rho \times A \times L \times g$ $\{\because \ L^2 = A = \text{Area}\}$

$$\therefore \quad F_e = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}} \qquad ...(12.13)$$

12.8.3 Euler's Number (E_u). It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as

$$E_u = \sqrt{\frac{F_i}{F_p}}$$

where F_p = Intensity of pressure × Area = $p \times A$

and $F_i = \rho A V^2$

$$\therefore \quad E_u = \sqrt{\frac{\rho A V^2}{p \times A}} = \sqrt{\frac{V^2}{p / \rho}} = \frac{V}{\sqrt{p / \rho}} \quad \text{...(12.14)}$$

12.8.4 Weber's Number (W_e). It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

Weber's Number, $$W_e = \sqrt{\frac{F_i}{F_s}}$$

where F_i = Inertia force = $\rho A V^2$

and F_s = Surface tension force

= Surface tension per unit length × Length = $\sigma \times L$

$$\therefore \quad W_e = \sqrt{\frac{\rho A V^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 \times V^2}{\sigma \times L}} \qquad \{\because \; A = L^2\}$$

$$= \sqrt{\frac{\rho L \times V^2}{\sigma}} = \sqrt{\frac{V^2}{\sigma / \rho L}} = \frac{V}{\sqrt{\sigma / \rho L}}. \qquad ...(12.15)$$

12.8.5 Mach's Number (M). Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. Mathematically, it is defined as

$$M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{F_i}{F_e}}$$

where $F_i = \rho A V^2$

and F_e = Elastic force = Elastic stress × Area

$= K \times A = K \times L^2$ $\{\because \; K = \text{Elastic stress}\}$

$$\therefore \quad M = \sqrt{\frac{\rho A V^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 \times V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}$$

But $\sqrt{\frac{K}{\rho}} = C$ = Velocity of sound in the fluid

$$\therefore \quad M = \frac{V}{C}. \qquad ...(12.16)$$

Model Laws

The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity.

1. Reynold’s Model

Models based on Reynolds’s Number includes:

a) Pipe Flow
b) Resistance experienced by Sub-marines, airplanes, fully immersed bodies

Model Laws

The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity.

1. Reynold’s Model
2. Froude Model Law

Froude Model Law is applied in the following fluid flow problems:

a) Free Surface Flows such as Flow over spillways, Weirs, Sluices, Channels etc.,
b) Flow of jet from an orifice or nozzle
c) Where waves are likely to formed on surface

Model Laws

The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity.

1. Reynold's Model
2. Froude Model Law
3. Euler Model Law

Euler Model Law is applied in the following cases:

a) Closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension is absent
b) Where phenomenon of cavitations takes place

Model Laws

The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity.

1. Reynold's Model
2. Froude Model Law
3. Euler Model Law
4. Weber Model Law

Weber Model Law is applied in the following cases:

a) Capillary rise in narrow passages
b) Capillary movement of water in soil
c) Capillary waves in channels
d) Flow over weirs for small heads

Model Laws

The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity.

1. Reynold's Model
2. Froude Model Law
3. Euler Model Law
4. Weber Model Law
5. Mach Model Law

Mach Model Law is applied in the following cases:

a) Flow of aero plane and projectile through air at supersonic speed ie., velocity more than velocity of sound
b) Aero dynamic testing, c) Underwater testing of torpedoes, and
d) Water-hammer problems

Reynold's Model Law

If the viscous forces are predominant, the models are designed for dynamic similarity based on Reynold's number.

$$[R_e]_m = [R_e]_p$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

$$t_r = \text{Time Scale Ratio} = \frac{L_r}{V_r}$$

Velocity, V = Length/Time ➔ T = L/V

$$a_r = \text{Acceleration Scale Ratio} = \frac{V_r}{t_r}$$

Acceleration, a = Velocity/Time ➔ L = V/T

Problem 6.15 *A pipe of diameter 1.5 m is required to transport an oil of sp. gr. 0.90 and viscosity 3×10^{-2} poise at the rate of 3000 litre/s. Tests were conducted on a 15 cm diameter pipe using water at 20°C. Find the velocity and rate of flow in the model. Viscosity of water at 20°C = 0.01 poise.*

(Delhi University, 1992)

Solution. Given :

Dia. of prototype, $D_p = 1.5$ m

Viscosity of fluid, $\mu_p = 3 \times 10^{-2}$ poise

Q for prototype, $Q_p = 3000$ lit/s $= 3.0$ m^3/s

Sp. gr. of oil, $S_p = 0.9$

$\therefore$ Density of oil, $\rho_p = S_p \times 1000 = 0.9 \times 1000 = 900$ kg/m^3

Dia. of the model, $D_m = 15$ cm $= 0.15$ m

Viscosity of water at 20°C $= .01$ poise $= 1 \times 10^{-2}$ poise or $\mu_m = 1 \times 10^{-2}$ poise

Density of water or $\rho_m = 1000$ kg/m^3.

For pipe flow, the dynamic similarity will be obtained if the Reynold's number in the model and prototype are equal

Hence using equation (6.17), $\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_P V_P D_P}{\mu_P}$ {For pipe, linear dimension is D}

$$\therefore \quad \frac{V_m}{V_P} = \frac{\rho_P}{\rho_m} \cdot \frac{D_P}{D_m} \cdot \frac{\mu_P}{\mu_m}$$

$$= \frac{900}{1000} \times \frac{1.5}{0.15} \times \frac{1 \times 10^{-2}}{3 \times 10^{-2}} = \frac{900}{1000} \times 10 \times \frac{1}{3} = 3.0$$

But

$$V_P = \frac{\text{Rate of flow in prototype}}{\text{Area of prototype}} = \frac{3.0}{\frac{\pi}{4}(D_P)^2} = \frac{3.0}{\frac{\pi}{4}(1.5)^2}$$

$$= \frac{3.0 \times 4}{\pi \times 2.25} = 1.697 \text{ m/s}$$

$$\therefore \quad V_m = 3.0 \times V_P = 3.0 \times 1.697 = \mathbf{5.091\ m/s.\ Ans.}$$

Rate of flow through model, $Q_m = A_m \times V_m = \frac{\pi}{4} (D_m)^2 \times V_m = \frac{\pi}{4} (0.15)^2 \times 5.091 \text{ m}^3\text{/s}$

$$= 0.0899 \text{ m}^3\text{/s} = 0.0899 \times 1000 \text{ lit/s} = \mathbf{89.9\ lit/s.\ Ans.}$$

Problems

1. Water flowing through a pipe of diameter 30 cm at a velocity of 4 m/s. Find the velocity of oil flowing in another pipe of diameter 10cm, if the conditions of dynamic similarity is satisfied between two pipes. The viscosity of water and oil is given as 0.01 poise and 0.025 poise. The specific gravity of oil is 0.8.

Froude Model Law

If the gravity force is predominant, the models are designed for dynamic similarity based on Froude number.

$$[F_e]_m = [F_e]_p \Rightarrow \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_P}} \Rightarrow V_r = \text{Velocity Scale Ratio} = \sqrt{L_r}$$

$$T_r = \text{Scale Ratio for Time} = \sqrt{L_r}$$

$$F_r = \text{Scale Ratio for Force} = L_r^{3}$$

$$T_r = \text{Scale Ratio for Acceleration} = 1$$

$$F_r = \text{Scale Ratio for Pressure Intensity} = L_r$$

$$Q_r = \text{Scale Ratio for Discharge} = L_r^{2.5}$$

$$P_r = \text{Scale Ratio for Power} = L_r^{3.5}$$

Velocity Ratio:

According to Froude Model

$$(F_e)_{model} = (F_e)_{prototype} \text{ or } \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_P}{\sqrt{g_P L_P}} \qquad \text{...(6.18)}$$

If the tests on the model are performed on the same place where prototype is to operate, then $g_m = g_P$ and equation (6.18) becomes as

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}} \qquad \text{...(6.19)}$$

or

$$\frac{V_m}{V_P} \times \frac{1}{\sqrt{\frac{L_m}{L_P}}} = 1$$

$$\frac{V_P}{V_m} = \sqrt{\frac{L_P}{L_m}} = \sqrt{L_r} \qquad \left\{\because \frac{L_P}{L_m} = L_r\right\}$$

where L_r = Scale ratio for length

(*a*) **Scale ratio for time**

$$\text{As time} = \frac{\text{Length}}{\text{Velocity}},$$

then ratio of time for prototype and model is

$$T_r = \frac{T_P}{T_m} = \frac{\left(\dfrac{L}{V}\right)_P}{\left(\dfrac{L}{V}\right)_m} = \frac{\dfrac{L_P}{V_P}}{\dfrac{L_m}{V_m}} = \frac{L_P}{L_m} \times \frac{V_m}{V_P} = L_r \times \frac{1}{\sqrt{L_r}} \qquad \left\{\because \ \frac{V_P}{V_m} = \sqrt{L_r}\right\}$$

$$= \sqrt{L_r}\,. \qquad \text{...(6.21)}$$

(*b*) **Scale ratio for acceleration**

$$\text{Acceleration} = \frac{V}{T}$$

$$\therefore \quad a_r = \frac{a_P}{a_m} = \frac{\left(\frac{V}{T}\right)_P}{\left(\frac{V}{T}\right)_m} = \frac{V_P}{T_P} \times \frac{T_m}{V_m} = \frac{V_P}{V_m} \times \frac{T_m}{T_P}$$

$$= \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} \qquad \left\{ \because \ \frac{V_P}{V_m} = \sqrt{L_r}, \frac{T_P}{T_m} = \sqrt{L_r} \right\}$$

$$= 1. \qquad \text{...(6.22)}$$

(*c*) **Scale ratio for discharge**

$$Q = A \times V = L^2 \times \frac{L}{T} = \frac{L^3}{T}$$

$$\therefore \qquad Q_r = \frac{Q_P}{Q_m} = \frac{\left(\frac{L^3}{T}\right)_P}{\left(\frac{L^3}{T}\right)_m} = \left(\frac{L_P}{L_m}\right)^3 \times \left(\frac{T_m}{T_P}\right) = L_r^{\,3} \times \frac{1}{\sqrt{L_r}} = L_r^{\,2.5} \qquad \text{...(12.23)}$$

(*d*) **Scale ratio for force**

$$\text{As Force} = \text{Mass} \times \text{Acceleration} = \rho L^3 \times \frac{V}{T} = \rho L^2 \,.\, \frac{L}{T} \,.\, V = \rho L^2 V^2$$

$$\therefore \quad \text{Ratio for force,} \qquad F_r = \frac{F_P}{F_m} = \frac{\rho_P L_P^2 V_P^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2.$$

If the fluid used in model and prototype is same, then

$$\frac{\rho_P}{\rho_m} = 1 \quad \text{or} \quad \rho_P = \rho_m$$

$$\text{and hence} \qquad F_r = \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2 = L_r^{\,2} \times \left(\sqrt{L_r}\right)^2 = L_r^{\,2} \,.\, L_r = L_r^{\,3}. \qquad \text{...(12.24)}$$

(*e*) **Scale ratio for pressure intensity**

As $$p = \frac{\text{Force}}{\text{Area}} = \frac{\rho L^2 V^2}{L^2} = \rho V^2$$

∴ Pressure ratio, $$p_r = \frac{p_P}{p_m} = \frac{\rho_P V_P^2}{\rho_m V_m^2}$$

If fluid is same, then $$\rho_P = \rho_m$$

∴ $$p_r = \frac{V_P^2}{V_m^2} = \left(\frac{V_P}{V_m}\right)^2 = L_r. \quad \text{...(12.25)}$$

(*f*) **Scale ratio for work, energy, torque, moment etc.**

$$\text{Torque} = \text{Force} \times \text{Distance} = F \times L$$

∴ Torque ratio, $$T_r^* = \frac{T_P^*}{T_m^*} = \frac{(F \times L)_P}{(F \times L)_m} = F_r \times L_r = L_r^3 \times L_r = L_r^4. \quad \text{...(12.26)}$$

(*g*) **Scale ratio for power**

As Power = Work per unit time

$$= \frac{F \times L}{T}$$

∴ Power ratio,

$$\rho_r = \frac{\rho_P}{\rho_m} = \frac{\frac{F_P \times L_P}{T_P}}{\frac{F_m \times L_m}{T_m}} = \frac{F_P}{F_m} \times \frac{L_P}{L_m} \times \frac{1}{\frac{T_P}{T_m}}$$

$$= F_r \,.\, L_r \,.\, \frac{1}{T_r} = L_r^{\,3} \,.\, L_r \,.\, \frac{1}{\sqrt{L_r}} = L^{3.5}. \qquad \text{...(12.27)}$$

Problems

1. In 1 in 40 model of a spillway, the velocity and discharge are 2 m/s and 2.5 m^3/s. Find corresponding velocity and discharge in the prototype

2. In a 1 in 20 model of stilling basin, the height of the jump in the model is observed to be 0.20m. What is height of hydraulic jump in the prototype? If energy dissipated in the model is 0.1kW, what is the corresponding value in prototype?

3. A 7.2 m height and 15 m long spillway discharges 94 m^3/s discharge under a head of 2m. If a 1:9 scale model of this spillway is to be constructed, determine the model dimensions, head over spillway model and the model discharge. If model is experiences a force of 7500 N, determine force on the prototype.

Problems

4. A Dam of 15 m long is to discharge water at the rate of 120 cumecs under a head of 3 m. Design a model, if supply available in the laboratory is 50 lps

5. A 1:50 spillway model has a discharge of 1.5 cumecs. What is the corresponding discharge in prototype?. If a flood phenomenon takes 6 hour to occur in the prototype, how long it should take in the model

Reference

Chapter 12

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