Advanced Queuing Models
Unit – V – Part – A (Two Marks)

(Non-Markovian Queue)

1. Consider a tandem queue with 2 independent Markovian Server. The Situation at Server 1 is just as in M/M/1 model. What will be the type of queue in Server 2? Why? Nov/Dec 2012

Soln output of M/M/1 Model is another (M/M/1) model.

2. State Jackson's theorem for an open network. Nov/Dec 2012

Soln A queuing network is said to be open if customers may enter from outside the system, circulate among the service nodes from service Service Centers or and then leave the System.

A queuing network is called Jackson open network if it satisfies

(i) with m nodes where each node has infinite queue

(ii) arrival rate follows Pisson process with rate of r_i

(iii) Service time at each Channel at node i are indep/ and exponentially distributed with parameter μ_i

Balace equation $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i p_{ij}$ is called traffic eqn or flow balance eqn in open Jackson network

(3) When a M/G/1 queuing model will become a classic (M/M/1) queuing model (May/Jun12)

Soln If the Service rate ~~is~~ exponential M/G/1 queuing model will become a classic M/M/1 queuing model.

(4) State Pollazcek - Khinchine formula for the average number of Customers in a (M/G/1) queuing model (May/Jun12)

Soln $L_s = \dfrac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho$ ——— (1)

$L_q = L_s - \dfrac{\lambda}{\mu} = L_s - \rho$

$L_q = \dfrac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$ $\quad W_q = \dfrac{L_q}{\lambda} = \dfrac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}$

$W_s = \dfrac{L_s}{\lambda} = \dfrac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} + \dfrac{1}{\mu}$

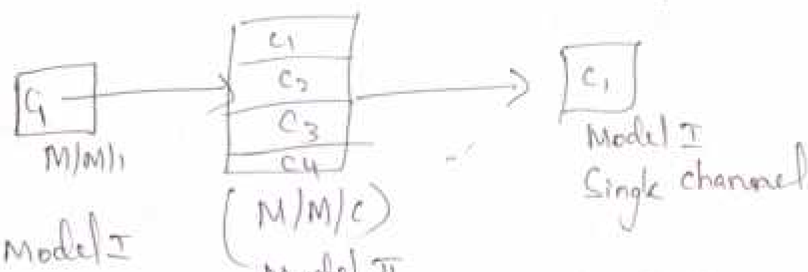
5) Write the Pollaczek - Khinchine formula (Nov / Dec 2011)

Soln :
$$L_s = E(N) = \lambda E(T) + \frac{\lambda^2 [Var(T) + E^2(T)]}{2[1 - \lambda E(T)]}$$

6) Define Series queues. (Nov / Dec 2011)

Soln : A series queuing model in which customers receive service at service facilities with no restriction on the waiting rooms capacity between facilities so that each facility has an infinite queue. Thus the series of facilities form a system of infinite queue in series.

(April / May 11)



7) Given that the service time is Erlang with parameter m and μ S.T P.K formula reduces to $L_s = m\rho + \frac{m(1+m)\rho^2}{2(1-m\rho)}$ (April / May 11)

Soln $L_s = \lambda E(T) + \frac{\lambda^2 [(E(T))^2 + V(T)]}{2[1 - \lambda E(T)]}$, $E(T) = \frac{m}{\mu}$, $Var(T) = \frac{m}{\mu^2}$

$$L_s = \frac{\lambda m}{\mu} + \frac{\lambda^2 \left(\frac{m^2}{\mu^2} + \frac{m}{\mu^2}\right)}{2\left(1 - \frac{\lambda m}{\mu}\right)} = m\rho + \frac{m(m+1)\rho^2}{2(1-m\rho)}$$

8. What is the Prob. that a Customer has to wait more than 15 min. to get his Service Completed in a M/M/1 queueing System, if $\lambda = 6$ per hour and $\mu = 10$ per hour.? (A.U ND 2004)

Soln. The probability that the waiting time of a Customer in the System exceeds

$$t = \int_t^{\infty} (\mu-\lambda)\, e^{-(\mu-\lambda)w}\, dw = e^{-(\mu-\lambda)t}$$

$$P\left(W_s > \frac{15}{60}\right) = e^{-(10-6)\frac{1}{4}} = e^{-1} = 0.3679$$

9. Give any two examples for Series queueing Situations. (April/May 2011)

Soln: Vechicle Servicing, Student Counselling

10. For $(M/M/1):(\infty/FIFO)$ model, write down the Little's formula. (A U A/M 2003)

Soln. (i) $E(N_s) = \frac{\lambda}{\mu-\lambda} = \lambda E(W_s)$ $\because E(N_s) = L$

(ii) $E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \lambda E(W_q)$ $\because E(N_q) = L_q$

(iii) $E(W_s) = E(W_q) + \frac{1}{\mu}$

(iv) $E(N_s) = E(N_q) + \frac{\lambda}{\mu}$

11. M/G/1 queueing System is Markovian ~~Commen~~ Comment on this Statement

Soln: M/G/1 queueing System is a non Markovian queue model. Since the Service time follows general distribution.

Unit - V - Part B (8 marks)

(Non - Markovian Queues)

1. Explain Non-Markovian queuing models and derive the Pollaczek - Khintchine formula. (May/Jun 2010) (16 marks)

Soln

Non - Markovian Queuing Model

The Models we have discussed so fare were easy because, the inter arrival and inter Service times were assumed to follow Exponential distributions with parameters λ and μ. Suppose if the arrivals and departures do not follow Poisson distribution then study of those models becomes difficult. But we can derive the formulas of a particular Non-Mark model $[M/G/1]:(\infty/GD)$ Model where M indicate the numbers of arrivals in time t which follows a Poisson process.

Pollaczek - Khinchine Formula

Let T be a Continuous random Variable. If L_s denotes number of Customers in the System

$L_s = E(N) =$ Average Number of Customers in the System

$$= \lambda E(T) + \frac{\lambda^2 \{Var(T) + (E(T))^2\}}{2\{1 - \lambda E(T)\}}$$

Proof: Here the Model $(M/G/1):(\infty/GD)$

M — denote the Number of arrivals in time t follows Poisson process

G → Service time follows a general distribution

1 — Single Server

GD — General queue Discipline

Arrival Rate λ is Poisson & Service Rate μ is not exponential (general distribution)

Let N and N' be the number of Customers on the system at time t and t+T, when two consecutive customers have just left the system after the Service.

Thus T is the random service time (continuous R.V)

M is a discrete R.V taking the values 0, 1, 2

Let $f(t)$, $E(T)$, $Var(T)$ — Variance of T.

$f(t)$ — PDF, $E(T)$ — Mean of T

Let M be the number of customers arriving in the system during the service time T

$$N' = \begin{cases} M & \text{if } N=0 \\ N-1+M & \text{if } N>0 \end{cases}$$ When M is a discrete R.V Taking values $0,1,2\ldots$

$\therefore N' = N+(M-1)+\delta$ — ① where $\delta = \begin{cases} 1 & \text{if } N=0 \\ 0 & \text{if } N>0 \end{cases}$

$E(N') = E(N) + E(M) - 1 + E(\delta) \longrightarrow$ ②

When the system is in steady state, the prob of number of customers in the system will be constant $\therefore E(N) = E(N')$ and $E(N^2) = E(N'^2)$ — ③

$E(M) - 1 + E(\delta) = 0 \Rightarrow \boxed{E(\delta) = 1 - E(M)}$ (since) $\longrightarrow$ ④

Squaring $N' = N+(M-1)+\delta$ on both sides, we get

$(N')^2 = N^2 + (M-1)^2 + \delta^2 + 2N(M-1) + 2(M-1)\delta + 2N\delta \longrightarrow$ ⑤

Now $\delta^2 = \delta$ (if $\delta = 0 \Rightarrow \delta^2 = 0$ if $\delta = 1 \Rightarrow \delta^2 = 1$) $\therefore \delta^2 = \delta$

Also $N\delta = \begin{cases} 0 \times 1 = 0 & \text{if } N=0 \\ N \times 0 = 0 & \text{if } N>0 \end{cases} \Rightarrow \boxed{N\delta = 0}$

$\therefore$ Eqn ⑤ becomes

$(N')^2 = N^2 + (M-1)^2 + \underline{\delta} + 2N(M-1) + 2(M-1)\delta + 0$ (Since $\delta^2 = \delta$, $N\delta = 0$)

$-2N(M-1) = N^2 - N'^2 + M^2 - 2M + 1 + \delta + 2M\delta - 2\delta$

$2E[N(1-M)] = E(N^2) - E(N'^2) + E(M^2) - 2E(M) + 1 + E[(2M-1)\delta]$

$$2E(N)E(1-M) = E(M^2) + E(2M-1)E(\delta) - 2E(M)+1$$

$$2E(N)E(1-M) = E(M^2) + \cancel{E(2M-1)}^{[2E(M)-1]} E(\delta) - 2E(M)+1 \quad (\text{sin } E(N^2) = E(N^2))$$

$$2E(N)[1-E(M)] = E(M^2) + [2E(M)-1]E(\delta) - 2E(M)+1$$

$$E(N) = \frac{E(M^2) + [2E(M)-1]E(\delta) - 2E(M)+1}{2[1-E(M)]}$$

$$= \frac{E(M^2) + [2E(M)-1][1-E(M)] - 2E(M)+1}{2[1-E(M)]} \quad \text{by (4)}$$

$$= \frac{E(M^2) + 2\cancel{E(M)} - 2E^2(M) \cancel{-1} + E(M) - \cancel{2E(M)} \cancel{+1}}{2[1-E(M)]}$$

$$E(N) = \frac{E(M^2) - 2E^2(M) + E(M)}{2[1-E(M)]} \longrightarrow (6)$$

Since the number of arrivals M in time T follows a Poisson process with parameter λ (say)

$$\therefore E(M/T) = \lambda T \qquad Var(M/T) = \lambda T$$

$$E(M^2/T) = \lambda^2 T^2 + \lambda T$$

$$E(M) = E(E(M/T)) = E(\lambda T) = \lambda E(T)$$

$(**)$

$$E(M^2) = E[E(M^2/T)] = E[\lambda^2 T^2 + \lambda T]$$

$$= \lambda^2 E(T^2) + \lambda E(T)$$

$$= \lambda^2 [Var(T) + E^2(T)] + \lambda E(T)$$

(6) becomes

$$L_s = E(N) = \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + \lambda E(T) - 2\lambda^2 E^2(T) + \lambda E(T)}{2[1 - \lambda E(T)]}$$

$$= \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + 2\lambda E(T) - 2\lambda^2 E^2(T)}{2[1 - \lambda E(T)]}$$

$$= \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + 2\lambda E(T)[1 - \lambda E(T)]}{2[1 - \lambda E(T)]}$$

$$= \frac{\lambda^2 Var(T) + \lambda^2 E^2(T)}{2[1 - \lambda E(T)]} + \frac{\cancel{2}\lambda E(T)\cancel{[1 - \lambda E(T)]}}{\cancel{2}\cancel{[1 - \lambda E(T)]}}$$

$$E(N) = \lambda E(T) + \frac{\lambda^2 [Var(T) + E^2(T)]}{2[1 - \lambda E(T)]}$$

Which is called Pollaczek-Khinchine formula, or P-K formula.

Note If $E(T) = \frac{1}{\mu}$ $V(T) = \sigma^2$ $\rho = \frac{\lambda}{\mu}$ then

$$L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

Note: In problem – Service time given exactly (or) mts is constant then apply P–K formula

2) a) An Automatic car wash facility operates with only one bay. Cars arriving according to a poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is constant and equal to 10 minutes determine L_s, L_q, W_s, W_q.

Soln Given $\lambda = 4$ cars/Hr $\mu = \frac{1}{10}$ per mts.

$= \frac{1}{10} \times 60 = 6$ car/Hr.

Given only one bay $\therefore s = 1$, capacity $= \infty$

Since the Service Rate is constant, the given problem comes under (M/G/1) Model

By using P–K formulae, $\sigma^2 = 0$ $\rho = \frac{\lambda}{\mu} = 0.667$

(i) $L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = \frac{0 + (0.667)^2}{2(1-0.667)} + 0.667$

$= 1.333$ Cars

ii) $L_q = L_s - \rho = 1.333 - 0.6667 = 0.6667$ Cars

iii) $W_s = \frac{L_s}{\lambda} = \frac{1.333}{4} \Rightarrow \boxed{W_s = 0.33\ Hr}$

iv) $W_q = \frac{L_q}{\lambda} = \frac{0.6667}{4} \Rightarrow \boxed{W_q = 0.165\ Hr}$

2b) An average of 120 students arrive each hour (inter arrival times are exponential) at the controller office to get their hall tickets. To complete the process, a candidate must pass through three counters. Each counters consists of a single server, service times: counter 1, 20 seconds, counter 2, 15 seconds and counter 3, 12 seconds. On the average how many students will be present in the controller's office. (May/Jue 2012)

Soln As there 3 stations (Q - series) all are comes under Model - I

$\lambda = 120/Hr$

Counter - I	Counter - II	Counter - III
$\mu_1 = \frac{1}{20}$ per second $= \frac{1}{20} \times 60$ per min $= 3$ per /minute $= 3 \times 60$ / Hr $\mu_1 = 180/Hr$	$\mu_2 = \frac{1}{15}$ per second $= \frac{1}{15} \times 60$ per mt $= 4 \times 60$ per Hr $\mu_2 = 240/Hr$	$\mu_3 = \frac{1}{12}$ per second $= \frac{1}{12} \times 60$ per min $= 5$ / mt $= 5 \times 60$ per Hr $\mu_3 = 300$ per Hr.

Average Students will be present in the Controller's office is given by $L_s = L_{s_1} + L_{s_2} + L_{s_3}$

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda} + \frac{\lambda}{\mu_3 - \lambda}$$

$$= \frac{120}{180-120} + \frac{120}{240-120} + \frac{120}{300-120} = \frac{120}{60} + \frac{120}{120} + \frac{120}{180}$$

$$= 2 + 1 + \frac{2}{3} = 3.6667 \text{ Students.}$$

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3) A repair facility Shared by a large Number of machines has two Sequential with respective Service rates of 2 per hour and 3/Hr the cumulative failure rate of all the machines is 1 per hour. Assuming that the system behaviour may be approximated by the 2-stage tandem queue, find (1) the average repair time including the waiting time (2) the prob/ that both the Service stations are idle (3) the bottle neck of the repair facility.

Soln This is Q-series. There are 2 Stations with Single server $\therefore$ It comes under (M/M/I)

Given failure rate $\lambda = 1$ per hr

$\mu_1 = 2/Hr$ (first station) $\mu_2 = 3/Hr$ (II station) put $j = 2$ in ① we get

$$\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22}$$

$$\lambda_2 = 5 + \frac{\lambda_1}{2}$$

$$-0.5\lambda_1 + \lambda_2 = 5 \quad \text{———} ③$$

$\therefore$ Solving ② & ③ (use calculator)

$\boxed{\lambda_1 = 6}$ $\boxed{\lambda_2 = 8}$

$\therefore$ Limiting probabilities in

$$P(m,n) = \left(\frac{\lambda_1}{\mu_1}\right)^m \left(1 - \frac{\lambda_1}{\mu_1}\right) \left(\frac{\lambda_2}{\mu_2}\right)^n \left(1 - \frac{\lambda_2}{\mu_2}\right)$$

$$= \left(\frac{3}{4}\right)^m \frac{1}{4} \left(\frac{4}{5}\right)^n \left(\frac{1}{5}\right)$$

$$= \frac{1}{20} \left(\frac{3}{4}\right)^m \left(\frac{4}{5}\right)^n$$

(ii) $L = L_s = L_{s_1} + L_{s_2}$

$$= \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{6}{8-6} + \frac{8}{10-8}$$

$$= \frac{6}{2} + \frac{8}{2} = 3 + 4 = 7 \text{ customers}$$

(iii) $W = W_s = \frac{L_s}{\lambda}$ when $\lambda = r_1 + r_2 = 4 + 5 = 9$

$$\therefore W_s = \frac{7}{9} = 0.77$$

$\boxed{W_s = 0.77 \text{ Hrs}}$

5) Derive P-K formula for the $(M/G/1):(GD/\infty)$ queuing model and hence deduce that with the constant service time the P-K formula reduces to $L_s = \rho + \frac{\rho^2}{2(1-\rho)}$ where $\mu = \frac{1}{E(T)}$, $\rho = \lambda/\mu$ (May/Jun 2012)

Soln (Derive P-k formula Question (1) of this Q. Bank)

$$E(N) = \frac{E(M^2) + E(M) - 2(E(M))^2}{2[1 - E(M)]}$$

$E(M/T) = \lambda T \qquad E(M^2/T) = (\lambda T)^2 + \lambda T$

$E(M) = E[E(M/T)] = E(\lambda T) = \lambda E(T)$

$= \left(\frac{\lambda}{\mu}\right)$

$E(M) = \rho \quad \therefore E(T) = 1/\mu$

$E(M^2) = E[E(M^2/T)] = E((\lambda T)^2 + \lambda T)$

$= \lambda^2 E(T^2) + \lambda E(T)$

$= \lambda^2 [\sigma^2 + E(T)^2] + \lambda/\mu$

$= \lambda^2 (\sigma^2 + 1/\mu^2) + \lambda/\mu = \lambda^2\sigma^2 + \frac{\lambda^2}{\mu^2} + \lambda/\mu$

$= \lambda^2\sigma^2 + \rho^2 + \rho$

$$\therefore E(N) = \frac{\lambda^2\sigma^2 + \rho^2 + \rho + \rho - 2\rho^2}{2(1-\rho)} = \frac{\lambda^2\sigma^2 + \rho^2 + 2\rho - 2\rho^2}{2(1-\rho)} = \frac{\lambda^2\sigma^2 + \rho^2 + 2\rho(1-\rho)}{2(1-\rho)}$$